Proving that a problem X is NP-hard requires several steps:

- Choose a problem Y that you already know is NP-hard.
- Describe an algorithm to solve Y, using an algorithm for X as a subroutine. Typically this algorithm has the following form: Given an instance of Y, transform it into an instance of X, and then call the magic black-box algorithm for X.
- Prove that your algorithm is correct. This almost always requires two separate steps:
  - Prove that your algorithm transforms "yes" instances of Y into "yes" instances of X.
  - Prove that your algorithm transforms "no" instances of Y into "no" instances of X. Equivalently: Prove that if your transformation produces a "yes" instance of X, then it was given a "yes" instance of Y.
- Argue that your algorithm for Y runs in polynomial time.

Proving that X is NP-Complete requires you to additionally prove that $X \in NP$ by describing a non-deterministic polynomial-time algorithm for X. Typically this is not hard for the problems we consider but it is not always obvious.

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1. Recall the following kCOLOR problem: Given an undirected graph G, can its vertices be colored with k colors, so that every edge touches vertices with two different colors?

   (a) Describe a direct polynomial-time reduction from 3COLOR to 4COLOR. *Hint:* Your reduction will take a graph G and output another graph G' such that G' is 4-colorable if and only if G is 3-colorable. You should think how an explicit 4-coloring for G' would enable you to obtain an explicit 3-coloring for G.

   (b) Prove that kCOLOR problem is NP-hard for any $k \geq 3$.

2. Describe a polynomial-time reduction from 3COLOR to SAT. Can you generalize it to reduce KCOLOR to SAT. *Hint:* Use a variable $x(v, i)$ to indicate that v is colored i and express the constraints using clauses in CNF form.

3. Let $G = (V, E)$ be a directed graph with edge lengths $\ell(e), e \in E$. The lengths can be positive or negative. The Zero-Length-Cycle Problem is to decide whether G has a cycle C of length *exactly* equal to 0. Prove that this problem is NP-Complete. *Hint: reduce Hamiltonian Path to Zero-Length-Cycle*