

1. Design a **two-tape** TM that computes the function $f(n) = \lceil n/2 \rceil$ when n is given in unary notation. More specifically, if the first tape initially holds $\$0^n$ with the first tape head scanning the first 0, and the second tape initially holds only $\$$ with the second tape head scanning the first blank to the right of the $\$$, then when started from the start state, your TM should halt with $\$0^{\lceil n/2 \rceil}$ on the *second* tape (and arbitrary contents on the first tape), and with second tape head scanning the symbol to the right of the left-edge marker $\$$. If the input is not of the specified form, it may behave arbitrarily.

Recall that a transition of a 2-tape machine is of the form $\delta(p, a, b) = (q, a', b', D_1, D_2)$, indicating that if the machine is in state p scanning the symbols a and b on tapes 1 and 2 respectively, then it writes the symbols a' and b' on tapes 1 and 2 respectively, moves the heads on tapes 1 and 2 in directions D_1 and D_2 respectively, and transitions to state q . (Here, each D_i is one of $\{L, R, S\}$.)

When specifying your TM, you can use variables to write your transition function more succinctly. For example, suppose that regardless of the symbol scanned on tape 2, if M is in state p and is scanning a 0 on tape 1, then it should transition to state q , write a 1 on tape 1, move right on tape 1, and leave tape 2 untouched, then you could write $\forall x \in \Gamma, \delta(p, 0, x) = (q, 1, x, R, S)$.

Think first of a high-level strategy and describe it carefully. Your TM should not need more than a few states, and when run on input $\$0^n$ it should halt in $O(n)$ steps.

2. Give a detailed description (but not the code) of a *multi-tape* TM that computes the function $f(n) = \lceil \log n \rceil$ when n is given in unary notation. In particular, if your TM is started with tape 1 holding $\$0^{2^k}$ (and all other tapes empty), it should halt with tape 1 holding $\$0^k$.

In your description, answer the following questions: How many tapes? What are their uses? What are the main phases of the computation? What are the states needed? What are their main functions? Your description can use phrases such as “copy tape 2 to tape 3”, or “erase the contents of tape 1,” or “move to the right on tape 1 until a 1 is encountered, and change it to 1’.” Specify all the symbols in your tape alphabet.

Give enough details that a TM programmer could implement it easily.

Hint: Can you use your 2-tape TM from problem 1?