“CS 374” Fall 2015 — Homework 10

Due Wednesday, December 2, 2015 at 10am

• • • Some important course policies • • •

- **You may work in groups of up to three people.** However, each problem should be submitted by exactly one person, and the beginning of the homework should clearly state the names and NetIDs of each person contributing.
- **You may use any source at your disposal**—paper, electronic, or human—but you ***must*** cite ***every*** source that you use. See the academic integrity policies on the course web site for more details. For all future homeworks, groups of up to three students will be allowed to submit joint solutions.
- **Submit your pdf solutions in Moodle.** See instructions on the course website and submit a separate pdf for each problem. Ideally, your solutions should be typeset in LaTex. If you hand write your homework make sure that the pdf scan is easy to read. Illegible scans will receive no points.
- **Avoid the Three Deadly Sins!** There are a few dangerous writing (and thinking) habits that will trigger an automatic zero on any homework or exam problem. Yes, we are completely serious.
  - Give complete solutions, not just examples.
  - Declare all your variables.
  - Never use weak induction.
- Unlike previous editions of this and other theory courses we are not using the “I don’t know” policy.

See the course web site for more information.

If you have any questions about these policies,
please don’t hesitate to ask in class, in office hours, or on Piazza.

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1. Reduce the Hamiltonian Path problem in Directed graphs to SAT via the following hints. Note that this means that you should come up with a polynomial-time algorithm that given a directed graph $G = (V, E)$ outputs a SAT formula φ such that φ is satisfiable iff G has a Hamiltonian Path. You can come up with your own reduction without looking at the outline below which is only one of several possible reductions.

   - Given directed graph $G = (V, E)$ use variables $x(v, i)$ for each v and $1 \le i \le n$ to indicate whether v is in the Hamiltonian path at position i.
   - Write constraints to ensure that for each $v \in V$ exactly one of $x(v, 1), x(v, 2), \dots, x(v, n)$ is set to 1. Here $n = |V|$.
   - Add variables $z(u, v)$ for each ordered pair (u, v). Write constraints such that $z(u, v)$ is 1 iff edge (u, v) is in E.
   - Write constraints to ensure that if $x(u, i)$ is 1 and $x(v, i + 1)$ is 1 then $z(u, v)$ is 1.
   - What is the size of the formula in your struction as a function of $|V|$ and $|E|$? By size of a SAT formula we mean the total sum of all the claus lengths.

   Justify the correctness of the reduction.

2. Suppose you are given a graph $G = (V, E)$ where V represents a collection of people and an edge between two people indicates that they are friends. You wish to break up V into at most k non-overlapping groups $V_1, V_2, \dots, V_k$ such that each group is very cohesive. One way to model cohesiveness is to insist that each pair of people in the same group should be friends; in other words they should form a clique. Given G and k show that the problem of deciding whether G can be partitioned into at most k cliques is NP-Complete. *Hint:* Consider a reduction from k-coloring.

3. MAX 2SAT is the decision problem where the input consists of a 2CNF formula φ and an integer L and the goal is to decide if there is an assignment to the variables such that at least L cluases of φ are satisfied. Note that 2SAT is polynomial-time solvable. Here we will see that MAX 2SAT is NP-Complete.

   (a) Consider two boolean variables x and y. Write a 2CNF formula that computes the function $\neg(x \wedge y)$.

   (b) Given a connected graph $G = (V, E)$ with n vertices and m edges, we want to compute its maximum size independent set. To this end, we define a boolean variable for every vertex of V. Describe how to write a 2CNF formula that is true if and only if the vertices that are assigned value 1 are all independent.

   (c) Given a graph G and an integer k, describe how to compute a 2CNF formula φ and a value L such that at least L clauses of φ can be satisfied if and only if there is an independent set in G of size k (or larger). To make things easy, you are allowed to duplicate the same clause in your formula as many times as you want. Naturally, the algorithm for computing this formula from G, k should work in polynomial time (and of course, you need to describe this algorithm). What is the value of L as a function of n, m and k where $n = |V|$ and $m = |E|$?

   (d) Using the preceding part prove that MAX 2SAT is NP-Complete.