CS 373 Summer 2011
Theory of Computation
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Unrecognizable Practice Soltions

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1. By contradiction, assume that L_1 is recognized by TM M_1. Let M_2 be a TM that decides L_2. $L = L_1 \setminus L_2 = L_1 \cap \overline{L_2}$ is recognizable as follows: on any input w, run M_1 on w. If it rejects, reject w, since $w \notin L_1$ implies $w \notin L$. If it accepts, run M_2 on w. If M_2 accepts, reject w.

   M_L: on input w
   run M_1 on w
   if accepts // $w \in L_1$
   run M_2 on w
   if accepts, **reject** w // $w \in L_1 \cap L_2 \Rightarrow w \notin L$
   else accept w // $w \in L_1 \cap \overline{L_2} \Rightarrow w \in L$
   else reject w //$w \notin L_1 \Rightarrow w \notin L$

   M_2 is a decider, so it will halt in finite time. If M_1 does not halt on w, then $w \notin L_1$ and $w \notin L$. Otherwise, the cases are computed as indicated in the comments above.

   Thus L is recognizable, which is a contradiction.

2. We provide a reduction for each problem.

   (a) We will show $E_{TM} \leq_m L_1$. Assume that there is a TM M_1 that recognizes L_1. Then the following TM recognizes E_{TM}:

   M_{ETM}: on input $\langle M \rangle$
   construct a TM N:
   N: on input x
   Move R once. Mark this cell.
   From now on until `STOP`, every R move is performed as two R moves,
   and every left move is two L moves
   Treat the marked cell as a new leftmost cell (i.e. L moves at marked cell will Stay).
   dovetail $(M, w_0), (M, w_1), (M, w_2), ...$
   where the sequence $w_0, w_1, w_2, ...$ is all strings in lexicographic order.
   if any accept, `STOP` doubling moves. Move R once and write a blank.
   accept x
   run M_1 on $\langle N \rangle$
   accept accordingly

Note that the first three lines of N guarantee that it only writes to even numbered positions of the tape until the accept case of the dovetail. Also note that N behaves the same no matter what input it is given ('MIDTERM2' or otherwise). If $\mathcal{L}(M) = \varnothing$, then N will never find a string that M accepts, and nothing will be written to an odd position on N's tape. If $\mathcal{L}(M) \neq \varnothing$, then N will find a string that M accepts, and something will written to an odd position of N's tape. Thus M_1 accepts $\langle N \rangle$ exactly when $\mathcal{L}(M) = \varnothing$.

(b) We will show $E_{TM} \leq_m \overline{L_{Pal}}$. Assume that there is a TM M_P that recognizes $\overline{L_{Pal}}$. Then the following TM recognizes E_{TM}:

```
M_ETM: on input ⟨M⟩
   construct a TM N:
      N: on input x
         dovetail (M, w_0), (M, w_1), (M, w_2), ...
               where the sequence w_0, w_1, w_2, ... is all strings in lexicographic order
         if any accept, accept x
   run M_P on ⟨N⟩
   accept accordingly
```

If $\mathcal{L}(M) = \varnothing$, then $\mathcal{L}(N) = \varnothing$ and $\langle N \rangle \in \overline{L_{Pal}}$ since it does not accept any palindromes. If $\mathcal{L}(M) \neq \varnothing$, then $\mathcal{L}(N) = \Sigma^*$ and $\langle N \rangle \notin \overline{L_{Pal}}$ since it accepts every palindrome. Thus M_P accepts $\langle N \rangle$ exactly when $\mathcal{L}(M) = \varnothing$.

(c) We will show $\overline{A_{TM}} \leq_m L_{ALL}$. Assume that there is a TM M_A that recognizes L_{ALL}. Then the following TM recognizes $\overline{A_{TM}}$:

```
M_CATM: on input ⟨M, w⟩
   construct a TM N:
      N: on input x
         run M on input w for |x| steps
         if M accepts, reject x
         if M rejects, accept x
         if M does neither within |x| steps, accept x
   run M_A on ⟨N⟩
   accept accordingly
```

If M rejects or loops on w, $\mathcal{L}(N) = \Sigma^*$ because N will accept whenever the simulation rejects or outputs nothing in the time limit. If M accepts w, $\mathcal{L}(N) \neq \Sigma^*$ because N will reject every string of length greater than or equal to the time of acceptance in M. Thus $\langle N \rangle \in L_{ALL}$ exactly when $\langle M, w \rangle \in \overline{A_{TM}}$.

(d) We will show $E_{TM} \leq_m \overline{L_{ALL}}$. Assume that there is a TM M_{CA} that recognizes $\overline{L_{ALL}}$. Then the following TM recognizes E_{TM}:

M_{ETM}: on input $\langle M \rangle$

construct a TM N:

> N: on input x
>
> **dovetail** $(M, w_0), (M, w_1), (M, w_2), ...$
>
> where the sequence $w_0, w_1, w_2, ...$ is all strings in lexicographic order
>
> **if** any accept, **accept** x

run M_{CA} on $\langle N \rangle$

accept accordingly

If $\mathcal{L}(M) = \varnothing$, then $\mathcal{L}(N) = \varnothing$ and $\langle N \rangle \in \overline{L_{ALL}}$ since it does not accept all strings. If $\mathcal{L}(M) \neq \varnothing$, then $\mathcal{L}(N) = \Sigma^*$ and $\langle N \rangle \notin \overline{L_{ALL}}$ since it accepts all strings. Thus M_{CA} accepts $\langle N \rangle$ exactly when $\mathcal{L}(M) = \varnothing$.