1 Introducing Finite Automata

1.1 Problems and Computation

Decision Problems

Decision Problems
Given input, decide "yes" or "no"

- *Examples:* Is x an even number? Is x prime? Is there a path from s to t in graph G?
- i.e., Compute a boolean function of input

General Computational Problem
In contrast, typically a problem requires computing some non-boolean function, or carrying out an interactive/reactive computation in a distributed environment

- *Examples:* Find the factors of x. Find the balance in account number x.
- In this course, we will study decision problems because aspects of computability are captured by this special class of problems

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What Does a Computation Look Like?

- Some code (a.k.a. *control*): the same for all instances
- The input (a.k.a. problem instance): encoded as a string over a finite alphabet
- As the program starts executing, some memory (a.k.a. *state*)
  - Includes the values of variables (and the "program counter")
  - State evolves throughout the computation
  - Often, takes more memory for larger problem instances
- But some programs do not need larger state for larger instances!

1.2 Finite Automata: Informal Overview

Finite State Computation

- *Finite state*: A fixed upper bound on the size of the state, independent of the size of the input
  - A sequential program with no dynamic allocation using variables that take boolean values (or values in a finite enumerated data type)

- If t-bit state, at most 2^t possible states

- Not enough memory to hold the entire input
  - "Streaming input": automaton runs (i.e., changes state) on seeing each bit of input

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An Automatic Door

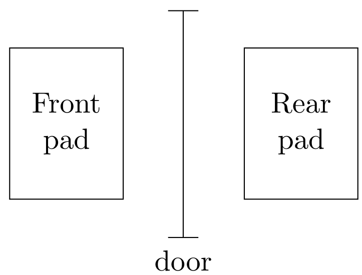


Figure 1: Top view of Door

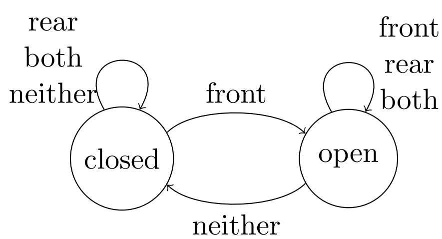


Figure 2: State diagram of controller

- *Input*: A stream of events `<front>`, `<rear>`, `<both>`, `<neither>` ...
- Controller has a single bit of state.

Finite Automata
Details

Automaton
A finite automaton has: Finite set of states, with *start/initial* and *accepting/final* states; *Transitions* from one state to another on reading a symbol from the input.

Computation
Start at the initial state; in each step, read the next symbol of the input, take the transition (edge) labeled by that symbol to a new state.

Acceptance/Rejection: If after reading the input w, the machine is in a final state then w is *accepted*; otherwise w is *rejected*.

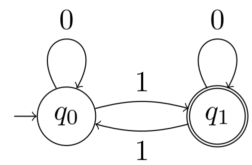


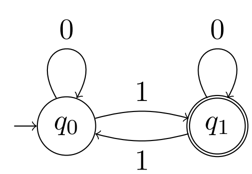
Figure 3: Transition Diagram of automaton

Conventions

- The initial state is shown by drawing an incoming arrow into the state, with no source.
- Final/accept states are indicated by drawing them with a double circle.

Example: Computation

- On input 1001, the computation is
  1. Start in state q_0. Read 1 and goto q_1.
  2. Read 0 and goto q_1.
  3. Read 0 and goto q_1.
  4. Read 1 and goto q_0. Since q_0 is not a final state 1001 is *rejected.*
- On input 010, the computation is
  1. Start in state q_0. Read 0 and goto q_0.
  2. Read 1 and goto q_1.
  3. Read 0 and goto q_1. Since q_1 is a final state 010 is *accepted.*



1.3 Applications

Finite Automata in Practice

- `grep`
- Thermostats
- Coke Machines
- Elevators
- Train Track Switches
- Security Properties
- Lexical Analyzers for Parsers

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2 Formal Definitions

2.1 Deterministic Finite Automaton

Defining an Automaton

To describe an automaton, we to need to specify

- What the alphabet is,
- What the states are,
- What the initial state is,
- What states are accepting/final, and
- What the transition from each state and input symbol is.

Thus, the above 5 things are part of the formal definition.

Deterministic Finite Automata
Formal Definition

Definition 1. A deterministic finite automaton (DFA) is $M = (Q, \Sigma, \delta, q_0, F)$, where

- Q is the finite set of states
- Σ is the finite alphabet
- $\delta : Q \times \Sigma \to Q$ "Next-state" transition function

	0	1
q_0	q_0	q_1
q_1	q_1	q_0

Figure 5: Transition Table representation

- $q_0 \in Q$ initial state
- $F \subseteq Q$ final/accepting states

Given a state and a symbol, the next state is "determined".

Formal Example of DFA

Example 2.

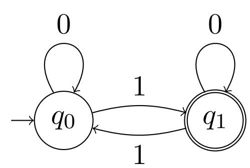

Figure 4: Transition Diagram of DFA

Formally the automaton is $M = (\{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{q_1\})$ where

$$\begin{array}{ll} \delta(q_0, 0) = q_0 & \delta(q_0, 1) = q_1 \\ \delta(q_1, 0) = q_1 & \delta(q_1, 1) = q_0 \end{array}$$

Computation

Definition 3. For a DFA $M = (Q, \Sigma, \delta, q_0, F)$, string $w = w_1 w_2 \cdots w_k$, where for each i $w_i \in \Sigma$, and states $q_1, q_2 \in Q$, we say $q_1 \xrightarrow{w}_M q_2$ if there is a sequence of states $r_0, r_1, \ldots r_k$ such that

- $r_0 = q_1$,
- for each i, $\delta(r_i, w_{i+1}) = r_{i+1}$, and
- $r_k = q_2$.

Definition 4. For a DFA $M = (Q, \Sigma, \delta, q_0, F)$ and string $w \in \Sigma^*$, we say M *accepts* w iff $q_0 \xrightarrow{w}_M q$ for some $q \in F$.

Useful Notation

Definition 5. For a DFA $M = (Q, \Sigma, \delta, q_0, F)$, let us define a function $\hat{\delta}_M : Q \times \Sigma^* \to \mathcal{P}(Q)$ such that $\hat{\delta}_M(q, w) = \{q' \in Q \mid q \xrightarrow{w}_M q'\}$.

We could say M accepts w iff $\hat{\delta}_M(q_0, w) \cap F \neq \emptyset$.

Proposition 6. *For a DFA $M = (Q, \Sigma, \delta, q_0, F)$, and any $q \in Q$, and $w \in \Sigma^*$, $|\hat{\delta}_M(q, w)| = 1$.*

Acceptance/Recognition

Definition 7. The *language accepted or recognized* by a DFA M over alphabet Σ is $\mathbf{L}(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\}$. A language L is said to be *accepted/recognized* by M if $L = \mathbf{L}(M)$.

2.2 Examples

Example I

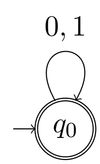

Figure 6: Automaton accepts all strings of 0s and 1s

Example II

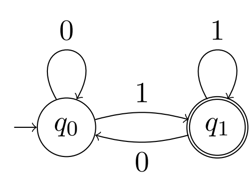

Figure 7: Automaton accepts strings ending in 1

Example III

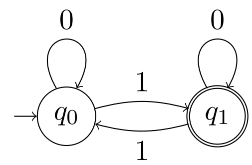

Figure 8: Automaton accepts strings having an odd number of 1s

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Example IV

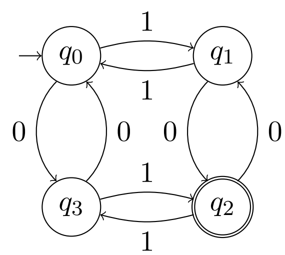

Figure 9: Automaton accepts strings having an odd number of 1s and odd number of 0s

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3 Designing DFAs

3.1 General Method

Typical Problem

Problem
Given a language L, design a DFA M that accepts L, i.e., $\mathbf{L}(M) = L$.

Methodology

- Imagine yourself in the place of the machine, reading symbols of the input, and trying to determine if it should be accepted.
- Remember at any point you have only seen a part of the input, and you don't know when it ends.
- *Figure out what to keep in memory.* It cannot be all the symbols seen so far: it must fit into a finite number of bits.

3.2 Examples

Strings containing 0

Problem
Design an automaton that accepts all strings over $\{0, 1\}$ that contain at least one 0.

Solution
What do you need to remember? Whether you have seen a 0 so far or not!

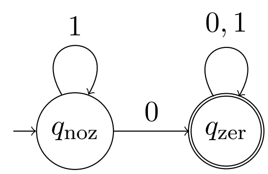


Figure 10: Automaton accepting strings with at least one 0.

Even length strings

Problem
Design an automaton that accepts all strings over $\{0, 1\}$ that have an even length.

Solution
What do you need to remember? Whether you have seen an odd or an even number of symbols.

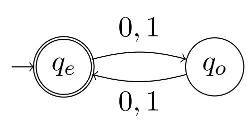


Figure 11: Automaton accepting strings of even length.

Pattern Recognition

Problem
Design an automaton that accepts all strings over $\{0, 1\}$ that have 001 as a substring, where u is a substring of w if there are w_1 and w_2 such that $w = w_1 u w_2$.

Solution
What do you need to remember? Whether you

- haven't seen any symbols of the pattern

- have just seen 0
- have just seen 00
- have seen the entire pattern 001

Pattern Recognition Automaton

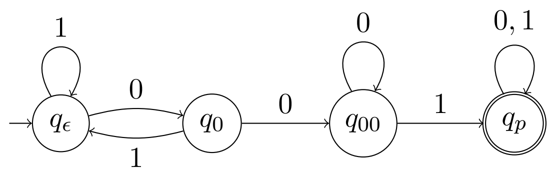
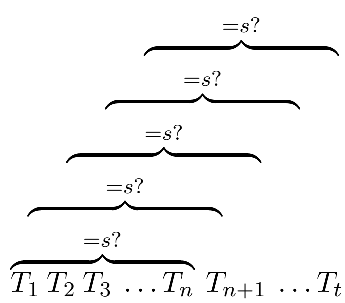


Figure 12: Automaton accepting strings having 001 as substring.

`grep` **Problem**

Problem
Given text T and string s, does s appear in T?

Naïve Solution



Running time $= O(nt)$, where $|T| = t$ and $|s| = n$.

`grep` **Problem**
Smarter Solution

Solution

- Build DFA M for $L = \{w \mid \text{there are } u, v \text{ } s.t. \text{ } w = usv\}$
- Run M on text T

Time = time to build M + $O(t)$!

Questions

- Is L regular no matter what s is?
- If yes, can M be built "efficiently"?

Knuth-Morris-Pratt (1977): Yes to both the above questions.

Multiples

Problem
Design an automaton that accepts all strings w over $\{0, 1\}$ such that w is the binary representation of a number that is a multiple of 5.

Solution
What must be remembered? The remainder when divided by 5.
How do you compute remainders?

- If w is the number n then $w0$ is $2n$ and $w1$ is $2n + 1$.
- $(a.b + c) \bmod 5 = (a.(b \bmod 5) + c) \bmod 5$
- *e.g.* $1011 = 11$ (decimal) $\equiv 1 \bmod 5$ $10110 = 22$ (decimal) $\equiv 2 \bmod 5$ $10111 = 23$ (decimal) $\equiv 3 \bmod 5$

Automaton for recognizing Multiples

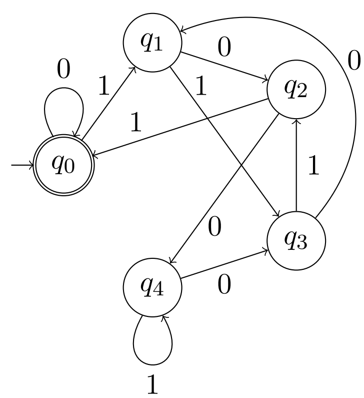


Figure 13: Automaton recognizing binary numbers that are multiples of 5.

A One k-positions from end

Problem

Design an automaton for the language $L_k = \{w \mid k\text{th character from end of } w \text{ is } 1\}$

Solution

What do you need to remember? The last k characters seen so far!

Formally, $M_k = (Q, \{0,1\}, \delta, q_0, F)$

- States $= Q = \{\langle w \rangle \mid w \in \{0,1\}^* \text{ and } |w| \leq k\}$
- $\delta(\langle w \rangle, b) = \begin{cases} \langle wb \rangle & \text{if } |w| < k \\ \langle w_2 w_3 \ldots w_k b \rangle & \text{if } w = w_1 w_2 \ldots w_k \end{cases}$
- $q_0 = \langle \epsilon \rangle$
- $F = \{\langle 1 w_2 w_3 \ldots w_k \rangle \mid w_i \in \{0,1\}\}$
