Problem Set 0

CS373 - Spring 2011

Due: Thursday Jan 27 at 2:00 PM in class (151 Everitt Lab)

Please follow the homework format guidelines posted on the class web page:

`http://www.cs.uiuc.edu/class/sp11/cs373/`

1. True/false

   [**Category**: Notation, **Points**: 10]

   Answer each of the following with true or false. Follow the notations in Sipser. We use $\{\dots\}$ to represent sets (not, for example, multisets). The symbol $P(S)$ denotes the set of all subsets of S (the power set of S). The symbol A represents an arbitrary non-empty set.

   D1) $\varnothing \in \{\varnothing\}$

   D2) $\varnothing \subseteq \{\{\varnothing\}\}$

   D3) $A \in P(P(A) \setminus \varnothing)$

   D4) $A \in P(P(A) \setminus A)$

   D5) $|P(P(P(\varnothing \cup A) \setminus A))| = 0$

   D6) $|P(P(A)) \setminus (\{A\} \cup A)| \neq 0$

   D7) $P(P(A)) \subset P(P(P(A)))$

   D8) $P(P(\varnothing)) \subset P(P(P(\varnothing)))$

   D9) $\{\varnothing\} \subseteq P(P(A) \setminus \varnothing)$

   D10) $\overline{A} = \varnothing \setminus A$

2. Bad induction

   [**Category**: Errors, **Points**: 10] A number $a \in \mathbb{N}$ *divides* another number $b \in \mathbb{N}$ if $\frac{b}{a} \in \mathbb{N}$. The following is a proof that every number in $\mathbb{N}$ divides every other number in $\mathbb{N}$. Let S be a finite subset of $\mathbb{N}$. We will show that every member of S must divide every other member of S by inducting over the size of S.

   Base case: $|S| = 1$. In this case, S contains a single member $a \in S$. Since $\frac{a}{a} = 1 \in \mathbb{N}$, every member of S divides every other member of S.

   Inductive hypothesis: If $S \subset \mathbb{N}$ and $|S| < n$, then all members of S divide each other.

   Inductive step: Let T be a subset of $\mathbb{N}$ containing n members. Since T is a countable set, we can index its members, meaning that $T = \{t_1, t_2, \dots, t_n\}$. Let $T_i = T \setminus \{t_i\}$. Since $|T_i| = n - 1$, all members of T_i divide each other by the inductive hypothesis. Let j, k, ℓ be distinct natural numbers less than or equal to n. Since $t_j \in T_\ell$ and $t_k \in T_\ell$, t_j must divide t_k and vice versa. Since the precise indexing of T is arbitrary, t_j and t_k could be any members of T, so all members of T must divide each other.

   Conclusion: Since $\forall x, y \in \mathbb{N}\ \exists S \subset \mathbb{N}$ such that $x \in S$ and $y \in S$, every natural number divides every other natural number.

   Explain in detail why this proof is wrong.

3. Uncountably infinite
   [**Category**: Proof, **Points**: 15]

   The set $\mathbb{Q}$ is the set of all rational numbers. Prove that the set of functions from $\mathbb{Q}$ to $\{1, 2, 3\}$ is uncountably infinite.

4. Induction
   [**Category**: Proof, **Points**: 15] Let S_n be a set of n pairwise non-parallel lines in the plane such that no three of them pass through a common point. These lines separate the plane into a number of (possibly unbounded) cells. Let $f(n)$ be the number of cells created by S_n. Use induction to determine a closed-form formula for $f(n)$.

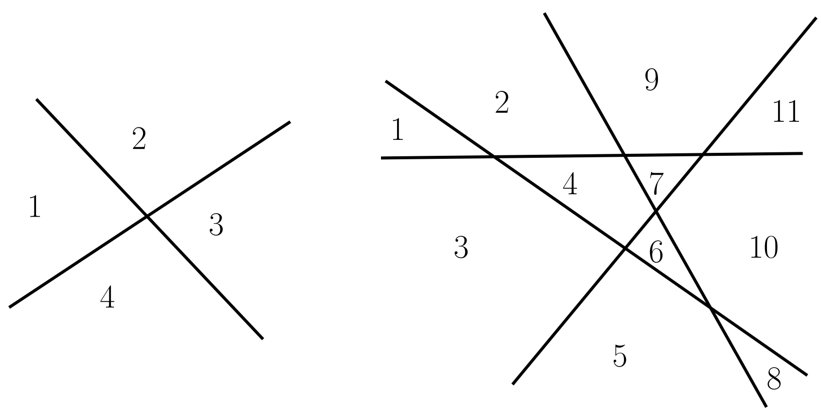

Figure 1: [left] Two lines separate the plane into four cells. Therefore, $f(2) = 4$. [right] Four lines separate the plane into eleven cells. Therefore, $f(4) = 11$.