CS 373: Intro to Theory of Computation
Spring 2010 **HW 10**, April, 2010

INSTRUCTIONS (read carefully)

- Print your name and netID here and netID at the top of each other page.

NAME:

NETID:

- It is wise to skim all problems and point values first, to best plan your time. If you get stuck on a problem, move on and come back to it later.
- Points may be deducted for solutions which are correct but excessively complicated, hard to understand, hard to read, or poorly explained.
- This is a closed book exam. No notes of any kind are allowed. Do all work in the space provided, using the backs of sheets if necessary. See the proctor if you need more paper.
- Please bring apparent bugs or unclear questions to the attention of the proctors.

Problem 1: True/False (14 points)

Completely write out "True" if the statement is necessarily true. Otherwise, completely write "False". Other answers (e.g. "T") will receive credit only if your intent is unambiguous. For example, "$x + y > x$" has answer "False" assuming that y could be 0 or negative. But "If x and y are natural numbers, then $x + y \geq x$" has answer "True". You do not need to explain or prove your answers.

1. If L_1 is context free and L_2 is not context free, then $L_1 L_2$ is not context free.

   False: ☐ **True**: ☐

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2. If L_1 is context free and L_2 is not context free, then $L_1 L_2$ is context free.

   False: ☐ **True**: ☐

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3. Every context free language is not regular.

   False: ☐ **True**: ☐

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4. If L_1 and L_2 are context free, then $L_1 \cap L_2$ is not context free.

   False: ☐ **True**: ☐

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5. If L is not context free, then it is not regular.

   False: ☐ **True**: ☐

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6. Let $\Sigma = \{a, b\}$ and $L = \{a^n w a^n \mid n \geq 1, w \in \Sigma^*\}$. L is not regular but is context free.

   False: ☐ **True**: ☐

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7. A non-deterministic TM can decide languages that a regular TM cannot decide.

   False: ☐ **True**: ☐

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8. For any $k > 1$, there is no language that is decided by a TM with k tapes, but is undecidable by any TM having $k - 1$ (or less) tapes.

   False: ☐ **True**: ☐

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9. If a language L is context-free then $\overline{L}$ is TM decidable.

   False: ☐ **True**: ☐

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10. The language $\overline{\mathrm{A}_{\mathsf{TM}}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ does not accept } w \right\}$ is TM recognizable.

    False: ☐ **True**: ☐

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11. It is possible for some undecidable language to be NP-Complete.

False: ☐ **True**: ☐

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12. Suppose L is TM recognizable but not TM decidable. Then any TM that recognizes L must fail to halt on an infinite number of strings.

False: ☐ **True**: ☐

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13. If VertexCover is in P, then HamiltonianPath is also in P.

False: ☐ **True**: ☐

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14. $3n^2 + 5n + 2 = O(\lg n + n^2/2)$

False: ☐ **True**: ☐

Problem 2: Classification (10 points)

For each language L described below, we have listed 2–3 language classes. Mark the most restrictive listed class to which L must belong. E.g. if L must always be context free and decidable (but not always regular) and we have listed "regular", "context-free" and "decidable", you must mark only "context-free".

(a) $L = \left\{ xw \;\middle|\; x, w \in \{\mathtt{a}, \mathtt{b}\}^* \text{ and } |x| = |w| \right\}$

☐ **Regular** ☐ **Context-free** ☐ **TM-Decidable**

(b) $L = \left\{ \mathtt{a}^i \mathtt{b}^j \mathtt{c}^k \mathtt{d}^m \;\middle|\; i + j + k + m \text{ is a multiple of } 13 \right\}$

☐ **Regular** ☐ **Context-free** ☐ **TM-Decidable**

(c) $L = \left\{ \langle w, M_1, M_2, \ldots, M_k \rangle \;\middle|\; \begin{array}{l} w \text{ is a string,} \\ k \text{ is an odd number larger than 2,} \\ \text{each } M_i \text{ is a } \mathsf{TM}, \\ \text{and a majority of the } M_i\text{'s accept } w \end{array} \right\}$

☐ **TM-Decidable** ☐ **TM-Recognizable** ☐ **Not TM recognizable**

(d) $L = \{x_1\#x_2\#\ldots\#x_n \mid x_i \in \{a, b\}^* \text{ for each } i \text{ and, for some } i, x_i \text{ is a palindrome}\}$.

☐ **Regular** ☐ **Context-free** ☐ **TM-Decidable**

(e) $L = \left\{ \langle G, D \rangle \;\middle|\; G \text{ is a } \mathsf{CFG}, D \text{ is a } \mathsf{DFA}, \text{and } L(G) \subseteq L(D) \right\}$

☐ **TM-Decidable** ☐ **TM-Recognizable** ☐ **Not TM recognizable**

Problem 3: Reduction (10 points)

Prove that L is undecidable where:

$$L = \{\langle M \rangle \mid M \text{ is a } \mathsf{TM} \text{ and accepts some sting of odd length}\}$$

You are not allowed to use Rice's Theorem in this problem (although you can adapt the proof of Rice's Theorem to this problem).

Problem 4: Decidability (10 points)

Is L decidable? Prove your claim.

$$L = \{\langle M \rangle \mid M \text{ is a } \mathsf{TM} \text{ and if we start } M \text{ with a blank input tape, then it will finally write some non-blank symbol on its tape.}\}$$

Problem 5: Palindrome (10 points)

Let $L = \left\{ \langle M \rangle \;\middle|\; M \text{ is a Turing machine and } M \text{ accepts at least one palindrome} \right\}$.

Show that L is TM-recognizable, i.e. explain how to construct a Turing machine that accepts $\langle M \rangle$ exactly when M accepts at least one palindrome. Of course, M might run forever on some input strings.

Problem 6: Grammar design (10 points)

Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$. Let

$$J = \left\{ w \;\middle|\; \#_{\mathtt{a}}(w) = \#_{\mathtt{b}}(w) \text{ or } \#_{\mathtt{b}}(w) = \#_{\mathtt{c}}(w) \right\},$$

where $\#_z(w)$ is the number of appearances of the character z in w. For example, the word $x = \mathtt{baccacbbcb} \in \mathsf{L}(J)$ since $\#_{\mathtt{a}}(x) = 2$, $\#_{\mathtt{b}}(x) = 4$, and $\#_{\mathtt{c}}(x) = 4$. Similarly, the word $y = \mathtt{abbccc} \notin \mathsf{L}(J)$ since $\#_{\mathtt{a}}(y) = 1$, $\#_{\mathtt{b}}(y) = 2$, and $\#_{\mathtt{c}}(y) = 3$.

Give a context-free grammar whose language is J. Be sure to indicate what its start symbol is. (Hint: First provide a CFG for the easier language $K = \left\{ w \in \{\mathtt{a}, \mathtt{b}\}^* \;\middle|\; \#_{\mathtt{a}}(w) = \#_{\mathtt{b}}(w) \right\}$ and modify it into the desired grammar.)

Problem 7: Non-CFLness (10 points)

1. Use pumping lemma for CFLs or the corollary to the pumping lemma for CFLs to prove that L is not regular:
$$L = \{w\#w \mid w \in \{0,1\}^*\}$$

2. Prove that L is not context free using closure properties:
$$L = \{a^n b^n w \mid n \geq 0, w \in \{c,d\}^*, |w| = n\}.$$
Note: $\{a^n b^n c^n \mid n \geq 0\}$ is the only language that you may assume we already knew that is not context free.

Problem 8: Normal Form (16 points)

Write this grammar in Chomsky Normal Form. Then use CYK algorithm to determine whether *ababb* is in the language of this grammar. (S is the start symbol)

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\begin{aligned}
S &\rightarrow ASA \mid aB \\
A &\rightarrow B \mid S \\
B &\rightarrow b \mid \epsilon
\end{aligned}
$$

Problem 9: Proof (10 points)

Prove fomally (preferably using induction) that all strings that the following grammar generates have even length. S is the start symbol.

$$S \rightarrow SB \mid aa$$
$$B \rightarrow bSBb \mid ab$$