Problem Set 0

Spring 2010

Due: Thursday Jan 28, 2:00pm, in class *before* the lecture.
Please follow the homework format guidelines posted on the class web page:
`http://www.cs.uiuc.edu/class/sp10/cs373/`

1. [**Category**: Notation, **Points**: 20]

   Answer each of the following my marking each with "**true**", "**false**", or "**wrong notation**." Follow the notations in Sipser. $\{\dots\}$ is used to represent sets and not multisets or anything else.

   D1) $\{a, b, c\} \cap \{d, e\} = \{\}$

   D2) $\{a, b, c\} \cap \{d, e\} = \{\emptyset\}$

   D3) $\{a, b, c\} \cup \{d, a, e\} = \{a, b, c, d, a, e\}$

   D4) $\{a, b, c\} \cup \{d, a, e\} = \{a, b, c, d, e\}$

   D5) $\{a, b, c\} \setminus \{a, d\} = \{b, c\}$

   D6) $\emptyset \in \{\emptyset, a, b, c\}$

   D7) $\emptyset \subseteq \{\emptyset, a, b, c\}$

   D8) $\emptyset \in \emptyset$

   D9) $a \subseteq \{\emptyset, a, b, c\}$

   D10) $\{a, c\} + \{c, b\} = \{a, b, c\}$

   D11) $\{a, b\} - \{b\} = \{a\}$

   D12) $\{a, a\} = \{a\}$

   D13) $\{\{a\}, \{a\}\} = \{a, a\}$

   D14) $a \in \{a, \{a\}, \{\{a\}\}\}$

   D15) $\{a\} \in \{a, \{a\}, \{\{a\}\}\}$

   D16) $\{\{\{a\}\}\} \subseteq \{a, \{a\}, \{\{a\}\}\}$

   D17) $\{\emptyset\} = \{\{\}\}$

   D18) $\{a, b\} \times \{c, d\} = \{(a, c), (b, d)\}$

   D19) $\{a, b\} \times \{c, d\} = \{c, d\} \times \{a, b\}$

   D20) $|\{a, b\} \times \{a, b\}| = 3$

Solution:

D1) $\{a, b, c\} \cap \{d, e\} = \{\}$ **true**

D2) $\{a, b, c\} \cap \{d, e\} = \{\emptyset\}$ **false**

D3) $\{a, b, c\} \cup \{d, a, e\} = \{a, b, c, d, a, e\}$ **true**

D4) $\{a, b, c\} \cup \{d, a, e\} = \{a, b, c, d, e\}$ **true**

D5) $\{a, b, c\} \setminus \{a, d\} = \{b, c\}$ **true**

D6) $\varnothing \in \{\varnothing, a, b, c\}$ **true**

D7) $\varnothing \subseteq \{\varnothing, a, b, c\}$ **true**

D8) $\varnothing \in \varnothing$ **false**

D9) $a \subseteq \{\varnothing, a, b, c\}$ **wrong notation**

D10) $\{a, c\} + \{c, b\} = \{a, b, c\}$ **wrong notation**

D11) $\{a, b\} - \{b\} = \{a\}$ **wrong notation (but we will also accept "true")**

D12) $\{a, a\} = \{a\}$ **true**

D13) $\{\{a\}, \{a\}\} = \{a, a\}$ **false**

D14) $a \in \{a, \{a\}, \{\{a\}\}\}$ **true**

D15) $\{a\} \in \{a, \{a\}, \{\{a\}\}\}$ **true**

D16) $\{\{\{a\}\}\} \subseteq \{a, \{a\}, \{\{a\}\}\}$ **true**

D17) $\{\varnothing\} = \{\{\}\}$ **true**

D18) $\{a, b\} \times \{c, d\} = \{(a, c), (b, d)\}$ **false**

D19) $\{a, b\} \times \{c, d\} = \{c, d\} \times \{a, b\}$ **false**

D20) $|\{a, b\} \times \{a, b\}| = 3$ **false**

2. [**Category**: Proof, **Points**: 16]

Professor Moriarty claims that he has a way of describing every real number between 0 and 1 using an English sentence (of finite length), i.e. for every real number r, there is an English sentence s that precisely describes r.

Prove that Professor Moriarty is wrong.

Note: Assume that a real number between 0 and 1 is of the form $0.a_1a_2a_3\ldots$, where each $a_i \in \{0, 1, \ldots 9\}$, i.e. is an infinite set of decimal points. This is not quite true, as $0.09999999\ldots$ is actually the same as $0.10000\ldots$, but ignore this subtlely for this question.

Solution:

The set of all finite English sentences, is after all a subset of strings over the English alphabet, and is hence countable.

The set of all real numbers is the set of all infinite sequences over $\{0, 1\}$, which is uncountable.

If there was a description of all real numbers using finite English sentences, then we can build a one-to-one correspondence between a subset of English sentences to the set of all reals (associate each real with the lexicographically smallest English sentence that describes it).

If a set A is infinite and countable, and B is infinite and uncountable, then there cannot be a 1-1 correspondence between them (if there was a 1-1 correspondence, say $f : A \to B$, then since A is countable, there is another 1-1 correspondence $g : \mathbb{N} \to B$, and the composition of these $f \circ g : \mathbb{N} \to B$ would be a 1-1 correspondence, which contradicts the fact that B is not countable).

Hence there cannot be a 1-1 correspondence between between a subset of English strings (which is countable) to the set of all reals (which is uncountable),

3. [**Category**: Proof, **Points**: 16]

   Prove that in a class with at least two students, there exist at least two students who have the same number of friends (assuming that friendship is a symmetric relation: if Jane is a friend of Venkatachalam, Venkatachalam is a friend of Jane too).

Solution:

Assume the class has n students. Each student can have x friends where $x \in \{0, 1, \cdots, n-1\}$. If no two students have the same number of friends, then for each $x \in \{0, \cdots, n-1\}$ there is exactly one student in class with x friends. But if there is a student with $n-1$ friends in class, then no student can have 0 friends; a contradiction.

4. [**Category**: Proof, **Points**: 16]

   A graph is said to be *non-isolating*, if every vertex has at least one edge incident on it.

   John guesses the following statement and proves it using induction.
   Guess: Every non-isolating graph is connected.
   proof: We use induction on the number of vertices of the graph to prove our statement.

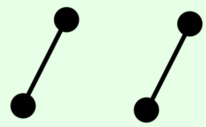
   Base-case: There is no non-isolating graph with one vertex. Moreover every 2-vertex non-isolating graph is trivially connected.

   Induction step: Assume the claim is true for all graphs with k vertices. Let G be a k-vertex non-isolating graph. By induction hypothesis G is connected. Now consider adding a new vertex u to G to give a non-isolating graph G' with $k+1$ vertices. Since G' is non-isolating, u must be connected to some other vertex of G', let's say it's connected to v. This implies that the $k+1$ vertex graph G' is connected (since we can reach from u to any other vertex x by going to v first and -since G is connected and both v and x are in G- then getting from v to x) and we are done.

   First show that John's guess is incorrect. Second identify clearly what is wrong with this inductive proof.

Solution:

The guess is incorrect since for example the following graph is non-isolating and is not connected.



The flaw in the induction is because *every* possible non-isolating $k+1$-vertex graph cannot be obtained by adding an additional vertex to a non-isolating k-vertex graph (For example the graph shown above). Therefore the mentioned proof just proves that for the particular graphs that could be obtained in that way the claim is true (but not for all the non-isolating graphs).

In general, this is a common mistake many people make. Let us assume that we are proving a property:
$P(n)$: Every graph of n vertices that satisfies the condition α, must satisfy β, for all $n \in \mathbb{N}$.

In (strong) induction, in the inductive step, we assume that $\forall i \leq n.P(n)$ holds, and we need to prove $P(n+1)$ holds. Then, in order to prove $P(n+1)$, we must show "every graph with $n+1$ vertices that satisfies α also satisfies β." To do this we must consider an *arbitrary* graph G with $n+1$ vertices, and using the assumption $\forall i \leq n.P(n)$, prove that the property holds for the graph G. In many proofs, we can *break down* the graph G to a smaller graph, use the induction hypothesis on the smaller graph, and show that the graph G satisfies the property. But we should *not* take a graph of n vertices (that satisfies α) and *add* a vertex to it; that is wrong because we do not know if all graphs with $n+1$ vertices that satisfy α can be obtained from an n-vertex graph that satisfies α, using such an operation.

5. [**Category**: Proof, **Points**: 16]

   12 players took part in a tennis tournament. Each pair of players played with each other exactly one time. There's no player who lost all his games (and there's no tie in tennis). Prove that there exist three players A, B and C, such that A defeated B, B defeated C and C defeated A.

Solution:

We will prove the existence of three players satisfying the condition, for *any* number of total players r, where $r \geq 3$ (instead of proving only for 12).

The first proof is by induction on the number of players.

Base: For $r = 3$, by checking all possible tournament results we can find out that the only possible option is A defeated B, B defeated C and C defeated A.

Inductive hypothesis: If any set of k players take part in the tournament (where $3 \leq k < r$, $r > 3$), there exist three players A, B and C, such that A defeated B, B defeated C and C defeated A.

Inductive step: Consider a tournament with r players. Consider some player B. Let $A_1, \ldots, A_n$ be the players who defeated B and $C_1, \ldots, C_m$ be the players who lost to B. There are two possible cases:

(a) There exists a player C_i who lost to all other players $C_j, j \neq i$. Since he also lost to B, there exists a player A_t who lost to C_i, otherwise C_i would have lost every game. Then the desired triple is A_t, B, C_i.

(b) No such C_i exists. By inductive hypothesis, there exists a desired triple among $C_1, \ldots, C_m$.

Alternative proof:
Assume there are n players, $n \geq 3$. Now, let us draw a graph with vertices as players, and directed edges $p \to q$ if p beats q in the tournament. Note that for every player p there is a player q such that $p \to q$ (since no player loses all games).

I can pick an arbitrary player p_1, take a successor of p_1, $p_1 \to p_2$, take a successor p_3 of p_2, $p1 \to p_2 \to p_3$, and build a longer and longer path until a vertex repeats. Hence there is always a directed cycle in this graph.

Now consider a directed cycle of the smallest length. I claim it has to be of length 3. It can't be of length 2 (since if p_1 beats p_2, then p_2 didn't beat p_1). Assume the smallest length cycle is of some length n, where $n > 3$. Let such a cycle be $p_1 \to p_2 \to p_3 \to \ldots \to p_n \to p_1$, where $n > 3$. Then consider the game between p_1 and p_3, which someone would have won, giving an edge from p_1 to p_3, or from p_3 to p_1. If $p_3 \to p_1$, then $p_1 \to p_2 \to p_3 \to p_1$ is a cycle of length 3, a contradiction. So $p_1 \to p_3$ must hold. But then $p_1 \to p_3 \to \ldots p_n \to p_1$ is a cycle of length $n-1$, again a contradiction. Hence the shortest cycle is of length 3, and we are done.

6. [**Category**: Proof, **Points**: 6+10]

Here is a theorem and a *formal* proof of it.

Theorem: *Let X and Y be two sets. Let $X \subseteq X \cap Y$. Then $X \subseteq Y$.*

Proof: Let $X \subseteq X \cap Y$. In order to show $X \subseteq Y$, we will show that if $s \in X$, then $s \in Y$. Let $s \in X$. Since $X \subseteq X \cap Y$, $s \in X \cap Y$. Therefore $s \in Y$.

In general, if you want to prove $X = Y$, it's good to break it up into two proofs: i.e., prove $X \subseteq Y$ and prove $Y \subseteq X$.

Now, prove the following theorems formally (using a similar style and level of detail as the proof above).

(a) Theorem: *Let X and Y be two sets, and let $X \cup Y = X \cap Y$. Then $X = Y$.*

(b) Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that for every $x, y \in \mathbb{N}$, if $x < y$, then $f(x) \geq f(y)$. Then there exists $s, t \in \mathbb{N}$ such that $s \neq t$ and $f(s) = f(t)$.

Write formal proofs. Don't wave hands. Don't say things like "it's obvious that", etc. If you are assuming a well known property, then state that property clearly.

Solution:

(a) Let $X \cup Y = X \cap Y$. We will prove $X = Y$ by proving $X \subseteq Y$ and proving $Y \subseteq X$.

To prove $X \subseteq Y$. Let $s \in X$. Then, since $X \cup Y = X \cap Y$, and $s \in X \cup Y$, $s \in X \cap Y$, Hence $s \in Y$. So $X \subseteq Y$.

Now let's prove $Y \subseteq X$. Let $s \in Y$. Then, since $X \cup Y = X \cap Y$, and $s \in X \cup Y$, $s \in X \cap Y$, Hence $s \in X$. So $Y \subseteq X$.

Hence $X = Y$.

(b)

Let S be the range of f. I.e. let $S = \{n \mid \exists m \in \mathbb{N}, f(m) = n\}$. Since S is a set of natural numbers, there must be a *least number*, say n_0 in S.

Now, let $m_0 \in \mathbb{N}$ be a number such that $f(m_0) = n_0$. Consider $f(m_0 + 1)$, and let it be x. We claim $x = n_0$.

First, since $m_0 < m_0 + 1$, $f(m_0) \geq f(m_0 + 1)$, i.e. $n_0 \geq x$. Since $x \in S$ (as x is in the range), and n_0 is the least number in S, $n_0 \leq x$. Hence $n_0 = x$.

So we have proved that there exists two distinct numbers, namely m_0 and $m_0 + 1$, such that $f(m_0) = f(m_0 + 1)$.