Problem Set 0

Spring 2010

Due: Thursday Jan 28, 2:00pm, in class *before* the lecture.

Please follow the homework format guidelines posted on the class web page:

`http://www.cs.uiuc.edu/class/sp10/cs373/`

1. [**Category**: Notation, **Points**: 20]

   Answer each of the following my marking each with "**true**", "**false**", or "**wrong notation**." Follow the notations in Sipser. $\{\ldots\}$ is used to represent sets and not multisets or anything else.

   D1) $\{a, b, c\} \cap \{d, e\} = \{\}$

   D2) $\{a, b, c\} \cap \{d, e\} = \{\varnothing\}$

   D3) $\{a, b, c\} \cup \{d, a, e\} = \{a, b, c, d, a, e\}$

   D4) $\{a, b, c\} \cup \{d, a, e\} = \{a, b, c, d, e\}$

   D5) $\{a, b, c\} \setminus \{a, d\} = \{b, c\}$

   D6) $\varnothing \in \{\varnothing, a, b, c\}$

   D7) $\varnothing \subseteq \{\varnothing, a, b, c\}$

   D8) $\varnothing \in \varnothing$

   D9) $a \subseteq \{\varnothing, a, b, c\}$

   D10) $\{a, c\} + \{c, b\} = \{a, b, c\}$

   D11) $\{a, b\} - \{b\} = \{a\}$

   D12) $\{a, a\} = \{a\}$

   D13) $\{\{a\}, \{a\}\} = \{a, a\}$

   D14) $a \in \{a, \{a\}, \{\{a\}\}\}$

   D15) $\{a\} \in \{a, \{a\}, \{\{a\}\}\}$

   D16) $\{\{\{a\}\}\} \subseteq \{a, \{a\}, \{\{a\}\}\}$

   D17) $\{\varnothing\} = \{\{\}\}$

   D18) $\{a, b\} \times \{c, d\} = \{(a, c), (b, d)\}$

   D19) $\{a, b\} \times \{c, d\} = \{c, d\} \times \{a, b\}$

   D20) $|\{a, b\} \times \{a, b\}| = 3$

2. [**Category**: Proof, **Points**: 16]

   Professor Moriarty claims that he has a way of describing every real number between 0 and 1 using an English sentence (of finite length), i.e. for every real number r, there is an English sentence s that precisely describes r.

   Prove that Professor Moriarty is wrong.

Note: Assume that a real number between 0 and 1 is of the form $0.a_1a_2a_3\ldots$, *where each* $a_i \in \{0, 1, \ldots 9\}$, *i.e. is an infinite set of decimal points. This is not quite true, as* $0.09999999\ldots$ *is actually the same as* $0.10000\ldots$, *but ignore this subtlely for this question.*

3. [**Category**: Proof, **Points**: 16]

   Prove that in a class with at least two students, there exist at least two students who have the same number of friends (assuming that friendship is a symmetric relation: if Jane is a friend of Venkatachalam, Venkatachalam is a friend of Jane too).

4. [**Category**: Proof, **Points**: 16]

   A graph is said to be *non-isolating*, if every vertex has at least one edge incident on it.

   John guesses the following statement and proves it using induction.
   Guess: Every non-isolating graph is connected.
   proof: We use induction on the number of vertices of the graph to prove our statement.

   Base-case: There is no non-isolating graph with one vertex. Moreover every 2-vertex non-isolating graph is trivially connected.

   Induction step: Assume the claim is true for all graphs with k vertices. Let G be a k-vertex non-isolating graph. By induction hypothesis G is connected. Now consider adding a new vertex u to G to give a non-isolating graph G' with $k+1$ vertices. Since G' is non-isolating, u must be connected to some other vertex of G', let's say it's connected to v. This implies that the $k+1$ vertex graph G' is connected (since we can reach from u to any other vertex x by going to v first and -since G is connected and both v and x are in G- then getting from v to x) and we are done.

   First show that John's guess is incorrect. Second identify clearly what is wrong with this inductive proof.

5. [**Category**: Proof, **Points**: 16]

   12 players took part in a tennis tournament. Each pair of players played with each other exactly one time. There's no player who lost all his games (and there's no tie in tennis). Prove that there exist three players A, B and C, such that A defeated B, B defeated C and C defeated A.

6. [**Category**: Proof, **Points**: 6+10]

   Here is a theorem and a *formal* proof of it.

   Theorem: *Let* X *and* Y *be two sets. Let* $X \subseteq X \cap Y$. *Then* $X \subseteq Y$.

   Proof: Let $X \subseteq X \cap Y$. In order to show $X \subseteq Y$, we will show that if $s \in X$, then $s \in Y$. Let $s \in X$. Since $X \subseteq X \cap Y$, $s \in X \cap Y$. Therefore $s \in Y$.

   In general, if you want to prove $X = Y$, it's good to break it up into two proofs: i.e., prove $X \subseteq Y$ and prove $Y \subseteq X$.

   Now, prove the following theorems formally (using a similar style and level of detail as the proof above).

(a) Theorem: *Let X and Y be two sets, and let $X \cup Y = X \cap Y$. Then $X = Y$.*

(b) Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that for every $x, y \in \mathbb{N}$, if $x < y$, then $f(x) \geq f(y)$. Then there exists $s, t \in \mathbb{N}$ such that $s \neq t$ and $f(s) = f(t)$.

Write formal proofs. Don't wave hands. Don't say things like "it's obvious that", etc. If you are assuming a well known property, then state that property clearly.