Problem Set 1

Spring 10

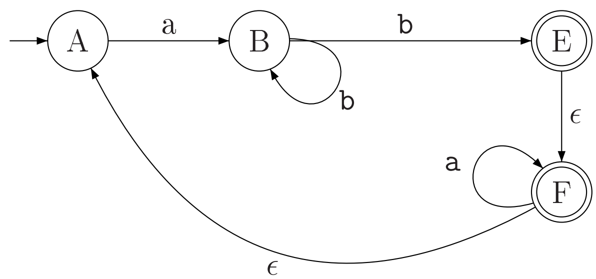
Due: Thursday Feb 11 in class before the lecture.

Please <u>follow</u> the homework format guidelines posted on the class web page:

`http://www.cs.uiuc.edu/class/sp10/cs373/`

1. NFA comprehension [**Category**: Comprehension, **Points**: 20]

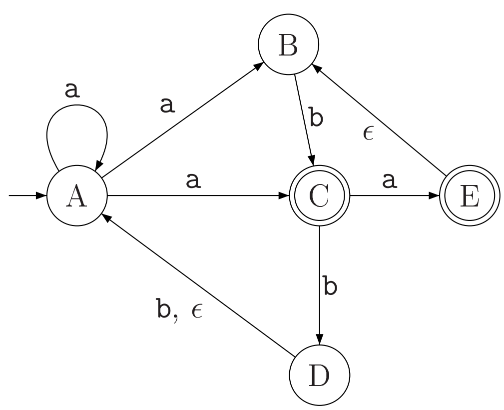
   Consider the following NFA M.



   (a) Give a regular expression that represents the language of M. Explain briefly why it is correct. (6 Points)

   (b) Recall the definition of an NFA accepting a string w (Sipser p. 54). Show formally that M accepts the string $w =$ abbab (6 Points)

   (c) Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Give the formal definition of the following NFA N (in tuple notation). Make sure you describe the transition function completely (for every state and every letter). (8 Points)

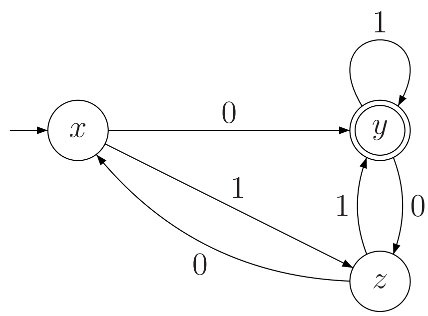


2. DFA Transformation [**Category**: Comprehension, **Points**: 20]

Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$, we define $\mathcal{T}(M)$ to be a new DFA $(Q', \Sigma, \delta', q_0', F')$ such that

$$
\begin{aligned}
&\forall q \in Q,\ N_q = \{\delta^*(q, ab) : a, b \in \Sigma\}\\
&Q' = \{(q, N_q) : q \in Q\} \cup \{T\}\\
&F' = \{(q, N_q) : q \in F\}\\
&q_0' = (q_0, N_{q_0})\\
&\forall q \in Q', \forall a \in \Sigma,\\
&\quad \delta'(q, a) = \begin{cases} T & q = T \\ \big(\delta(p, a), N_{\delta(p,a)}\big) & \exists p \in Q,\ q = (p, N_p) \text{ and } N_{\delta(p,a)} \subseteq \{\delta(r, a) : r \in N_p\} \\ T & \text{otherwise} \end{cases}
\end{aligned}
$$

Let M be the following DFA. Draw $\mathcal{T}(M)$ (label the states).



3. Language Projection [**Category**: Proof, **Points**: 20]

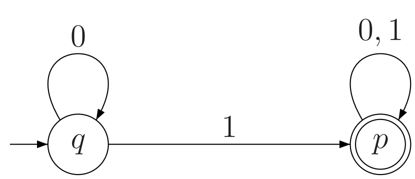
Let's define $w \downarrow \Sigma$ to be a word w', such that w' is equal to w with all symbols not in Σ removed. For example, $abcdbdcad \downarrow \{a, b, c\} = abcbca$.
Let L_1 be a regular language over the alphabet Σ_1 and L_2 be a regular language over the alphabet Σ_2.

Prove that $L = \{w \in (\Sigma_1 \cup \Sigma_2)^* \mid (w \downarrow \Sigma_1) \in L_1 \ \wedge\ (w \downarrow \Sigma_2) \in L_2\}$ is also regular.
Note: Σ_1 and Σ_2 may have common symbols.

4. Language of a DFA [**Category**: Proof, **Points**: 20]

First have a look at the following claim and its formal proof. The proof uses induction. You may think that since the claim is an easy fact you don't need such a heavy technique for proving it and in fact you are right! We could avoid induction and build a much easier proof for the claim. The reason that we have applied induction to prove this claim is to introduce this technique to you.

Claim: The language of the DFA D below is $A = \{0^n 1x : x \in \{0, 1\}^*, n \geq 0\}$.



Proof: Let $L(p)$ represent the set of all strings that if we feed them to the DFA D, then D will stop in state p. Similarly define $L(q)$ for state q. Note that since p is the only final state, we have $L(D) = L(p)$. Instead of proving the Claim directly, we will introduce a stronger claim and we will prove that stronger claim using induction (and this stronger claim is easier to attack using induction).

The Stronger Claim: $L(q) = C = \{0^n : n \geq 0\}$ and $L(p) = A = \{0^n 1x : x \in \{0,1\}^*, n \geq 0\}$.

Note that the stronger claim asks for everything in the previous old claim and also asks for something more; this is why sometimes it is called *overloaded claim.*

Proof of the Stronger Claim: Let B_k represent the set of all binary strings of length at most k. Using induction on k, we will prove that for every value of k, we have $L(q) \cap B_k = C \cap B_k$ and $L(p) \cap B_k = A \cap B_k$ (as an easy exercise, please justify for yourself that if we prove this, then we have proved the stronger claim).

Base case: When $k = 0$. We have $B_0 = \{\epsilon\}$. When we feed ϵ to D, it stops in state q and therefore $L(q) \cap B_0 = \{\epsilon\}$ and $L(p) \cap B_0 = \varnothing$. It is trivial to see that $C \cap B_0 = \{\epsilon\}$ and $A \cap B_0 = \varnothing$. Therefore we have $L(q) \cap B_0 = C \cap B_0$ and $L(p) \cap B_0 = A \cap B_0$.

Inductive Step: Assume that for some $k \geq 0$ we have $L(q) \cap B_k = C \cap B_k$ and $L(p) \cap B_k = A \cap B_k$, then we prove that $L(q) \cap B_{k+1} = C \cap B_{k+1}$ and $L(p) \cap B_{k+1} = A \cap B_{k+1}$.

First we prove $L(q) \cap B_{k+1} = C \cap B_{k+1}$. Since from induction hypothesis we know $L(q) \cap B_k = C \cap B_k$, we just need to show that $L(q) \cap \{0,1\}^{k+1} = C \cap \{0,1\}^{k+1}$ (justify this for yourself). Let $x \in L(q) \cap \{0,1\}^{k+1}$, write $x = x'a$ where $x' \in B_k$ and $a = 0$ or 1. Since $x \in L(q)$, we have $q = \delta^*(q,x) = \delta(\delta^*(q,x'),a)$. From this equation we have $\delta^*(q,x') = q$ and $a = 0$ (Why?). Since $\delta^*(q,x') = q$ by definition of $L(q)$ we have $x' \in L(q)$, and since $x' \in B_k$ we have $x' \in L(q) \cap B_k$. Since by induction hypothesis $L(q) \cap B_k = C \cap B_k$, we have $x' \in C \cap B_k$, and since we know that x' is of length k, we have $x' = 0^k$. But this means that $x = x'a = 0^k 0 = 0^{k+1}$. Since x was an arbitrary member of $L(q) \cap \{0,1\}^{k+1}$, we have $L(q) \cap \{0,1\}^{k+1} = \{0^{k+1}\}$. It is also trivial to see that $C \cap \{0,1\}^{k+1} = \{0^{k+1}\}$, therefore we have proved that $L(q) \cap \{0,1\}^{k+1} = C \cap \{0,1\}^{k+1}$.

Now we prove that $L(p) \cap B_{k+1} = A \cap B_{k+1}$ in a similar way. Since from induction hypothesis we know $L(p) \cap B_k = A \cap B_k$, we just need to show that $L(p) \cap \{0,1\}^{k+1} = A \cap \{0,1\}^{k+1}$ (again justify this for yourself). Let $x \in L(p) \cap \{0,1\}^{k+1}$, write $x = x'a$ where $x' \in B_k$ and $a = 0$ or 1. Since $x \in L(p)$, we have $p = \delta^*(q,x) = \delta(\delta^*(q,x'),a)$. From this last equation we have that either $\delta^*(q,x') = q$ and $a = 1$, or $\delta^*(q,x') = p$ and $a = 0$ or 1 (why?).

Case1: When $\delta^*(q,x') = q$ and $a = 1$. From definition of $L(q)$ we have that $x' \in L(q)$ and since $|x'| = k$ we have $x' \in L(q) \cap B_k$. By the induction hypothesis $L(q) \cap B_k = C \cap B_k$ and therefore $x' \in C \cap B_k$. Therefore $x' = 0^k$ and $x = x'a = 0^k 1 \in A \cap B_{k+1}$.

Case2: When $\delta^*(q,x') = p$ and $a = 0$ or 1. By definition of $L(p)$ we have $x' \in L(p)$ and hence $x' \in L(p) \cap B_k$. By induction hypothesis we have $L(p) \cap B_k = A \cap B_k$

and therefore $x' \in A \cap B_k$ and hence $x' = 0^n 1y$ for some $n \geq 0$ and $y \in \{0,1\}^*$ (such that $n + 1 + |y| = k$). Hence $x = x'a = 0^n 1ya \in A \cap B_{k+1}$.

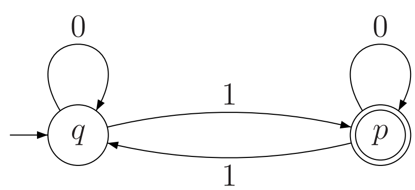
So up to this point we have proved that $L(p) \cap B_{k+1} \subseteq A \cap B_{k+1}$. Now we prove that $A \cap B_{k+1} \subseteq L(p) \cap B_{k+1}$. Let $x \in A \cap B_{k+1}$ we have $x = 0^n 1y$ for some $n \geq 0$ and $y \in \{0,1\}^*$ (such that $n + 1 + |y| = k + 1$). Now:

$$\delta^*(q, x) = \delta^*(q, 0^n 1y) = \delta^*(\delta^*(q, 0^n), 1y) = \delta^*(q, 1y) = \delta^*(\delta(q, 1), y) = \delta^*(p, y) = p$$

Therefore $x \in L(p)$ and since already $x \in B_{k+1}$, we have $x \in L(p) \cap B_{k+1}$, therefore $A \cap B_{k+1} \subseteq L(p) \cap B_{k+1}$. So we have proved that $A \cap B_{k+1} = L(p) \cap B_{k+1}$ and the induction is complete. □

Now prove formally that the language of the following DFA is:

$$\{x \in \{0,1\}^* : x \text{ has odd number of 1's}\}$$



5. Extra Credit/Honors [**Category**: Proof, **Points**: 20]

   Let L be a regular language. Show that $L' = \{w : ww^R \in L\}$ is regular.