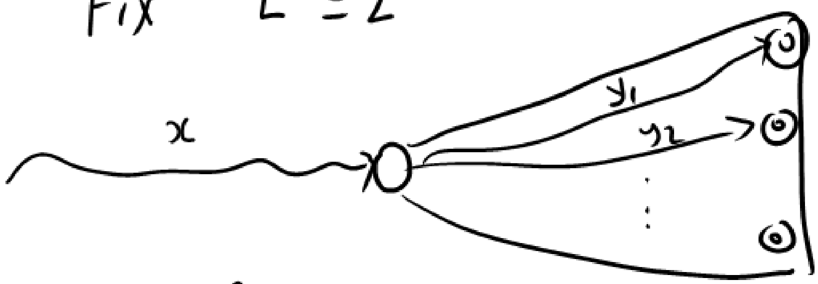
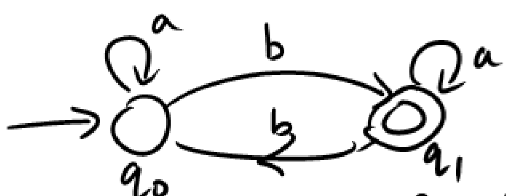
Lecture #9:

Myhill-Nerode theorem

Suffix languages

Fix $L \subseteq \Sigma^*$





Role of q_0 : accept $L_{q_0} = \{ w \mid w$ has an odd number of b's $\}$

Role of q_1 : accept $L_{q_1} = \{ w \mid w$ has an even number of b's $\}$.

Suffix language

Fix $L \subseteq \Sigma^*$. $\quad x \in \Sigma^*$

$$[\![L/x]\!] = \{ y \mid xy \in L \}$$

Eg. $\quad L = 0^* 1^*$

$$[\![L/000]\!] = 0^* 1^*$$

$$[\![L/1]\!] = 1^*$$

Let L be a regular language.

Then there is a DFA for L, say with n states.

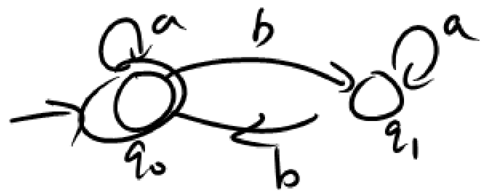
States $\{q_0, q_1 \dots q_{n-1}\}$

$L_{q_0}, L_{q_1}, \dots L_{q_{n-1}}$

I claim that the number of suffix languages is at most n

Why $[\![L/x]\!] = L_q$ where $\delta^*(q_0, x) = q$

$\Rightarrow$ So the number of suffix languages is at most the number of states.



$$L = a^*(a^* b\, a^* b)^*$$

$$L_{q_0} = L$$

$$L_{q_1} = a^* b\, (a^* b a^* b)^*$$

$[\![L/\epsilon]\!] = L$

$[\![L/a]\!] = L$

$[\![L/b]\!] = L_{q_1}$

$[\![L/bb]\!] = L$

$[\![L/ba]\!] = L_{q_1}$

$\forall x$ $[\![L/x]\!]$ is either L or L_{q_1}

We will show is that L is regular iff the number of suffix languages is finite.

($\Rightarrow$) Already proved.

$L = \{ a^n b^n \mid n \in \mathbb{N} \}$

$[\![L/a]\!] = \{ a^{n-1} b^n \mid n \in \mathbb{N} \}$

$[\![L/aa]\!] = \{ a^{n-2} b^n \mid n \in \mathbb{N} \}$

$\vdots$

$[\![L/a^i]\!] = \{ a^{n-i} b^n \mid n \in \mathbb{N} \}$

If there was a DFA for L, there would be a finite no. of suffix lang. But there isn't. Hence L is not regular.

So summarize.

Every state q $\longmapsto$ Suffix lang L_q.

But two state q and q' could have the same language.

The "role" of a state q is simply to accept L_q.

- $\varepsilon \in [\![L/x]\!]$ iff $x \in L$.
- Let $[\![L/x]\!] = [\![L/y]\!]$

Then $[\![L/xa]\!] = [\![L/ya]\!]$

$$
\begin{aligned}
z \in [\![L/xa]\!] \quad & \text{iff} \quad xaz \in L \\
& \text{iff} \quad az \in [\![L/x]\!] \\
& \text{iff} \quad az \in [\![L/y]\!] \\
& \text{iff} \quad yaz \in L \\
& \text{iff} \quad z \in [\![L/ya]\!]
\end{aligned}
$$

Lemma. Let $L \subseteq \Sigma^*$.

Let $\mathcal{C}(L)$ = set of all suffix languages of L

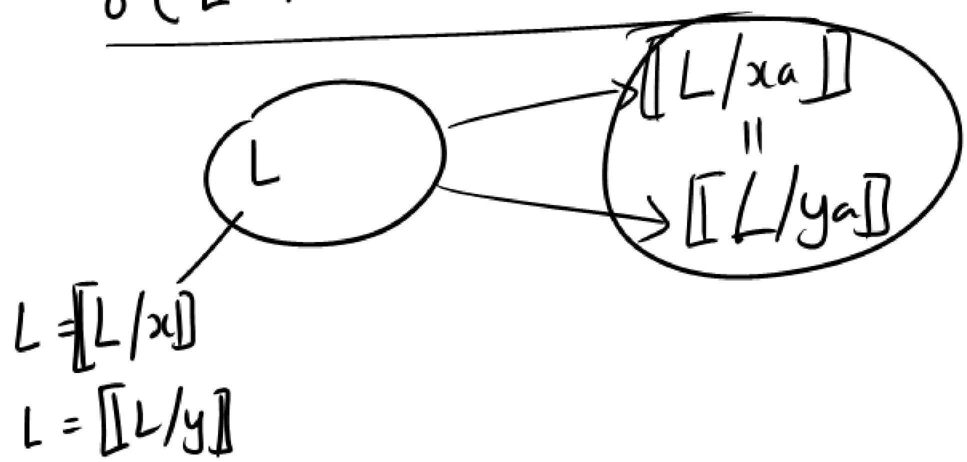
$= \{ [\![L/x]\!] \mid x \in \Sigma^* \}$

If $\mathcal{C}(L)$ is a finite set, then L is regular

Proof. Let $\mathcal{C}(L) = \{ [\![L/x_1]\!], [\![L/x_2]\!], \ldots [\![L/x_k]\!] \}$.

$$
\begin{aligned}
DFA &= (Q, \Sigma, \delta, q_0, F) \\
Q &= \{ [\![L/x_1]\!], [\![L/x_2]\!], \ldots [\![L/x_k]\!] \} \\
q_0 &= [\![L/\epsilon]\!] = L. \\
F &= \{ [\![L/x]\!] \mid \varepsilon \in [\![L/x]\!] \} \quad x \in L
\end{aligned}
$$

$$\delta([\![L/x]\!], a) = [\![L/xa]\!]$$



Claim <u>DFA A accepts L.</u>

<u>Proof by induction</u>

<u>Claim</u> After reading a word x, DFA A will be in state $[\![L/x]\!]$

If $x = \epsilon$, A is in its initial state, $[\![L/\epsilon]\!]$

If $x = wa$, A after reading w is in state $[\![L/w]\!]$

$$[\![L/w]\!] \xrightarrow{a} [\![L/wa]\!]$$

So after reading $x = wa$, A would be in state $[\![L/x]\!]$.

$x \in L$ iff $\epsilon \in [\![L/x]\!]$

iff $[\![L/x]\!]$ is a final state

~~iff $\delta^*($~~

$\delta^*(q_0, x) = [\![L/x]\!]$

$x \in L(A)$ iff $\delta^*(q_0, x) \in F$

iff $[\![L/x]\!] \in F$

iff $x \in L$

$L = 0^* 1^*$

$[\![L/\epsilon]\!] = L = 0^* 1^*$

$[\![L/0]\!] = L = 0^* 1^*$

$[\![L/1]\!] = 1^*$

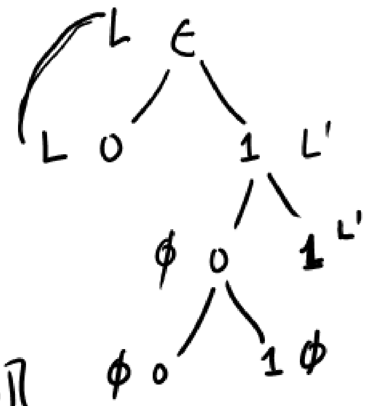
$[\![L/00]\!] = [\![L/\epsilon 0]\!] = [\![L/0]\!]$

$[\![L/10]\!] = \phi$

$[\![L/11]\!] = 1^*$

$[\![L/100]\!] = \phi$

$[\![L/101]\!] = \phi$



$\ell(L) = \{0^*1^*, 1^*, \phi\}$