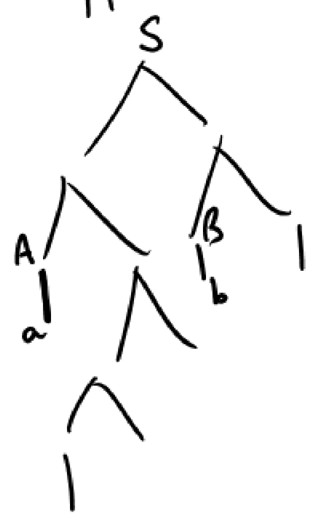
- Membership problem for CFGs
- CYK algorithm : $O(n^3)$ membership algm.
- Closure properties of CFLs
- RegLang $\subsetneq$ CFL $\subseteq$ Decidable Lang.

Chomsky Normal Forms

$A \rightarrow BC$ $\quad \| \quad S \rightarrow \epsilon$

$A \rightarrow a$



w , $|w| = n$

$S \Rightarrow^* w$

Involve generating $n-1$ more nonterminals (using rules $A \rightarrow BC$)

Involve convert nonterminals to terminals using n derivations (using rule $A \rightarrow a$)

If G is in CNF, and $S \Rightarrow^k w$ and $|w| = n$, then $k = 2n-1$.

Proof If $A \Rightarrow^r \alpha$ then $\#NT(\alpha) + 2\#T(\alpha) = r+1$

Base case $r=0$. $\alpha = A$

$\#NT(\alpha) = 1 \quad \#T(\alpha) = 0$

$r+1 = 1$ ✓

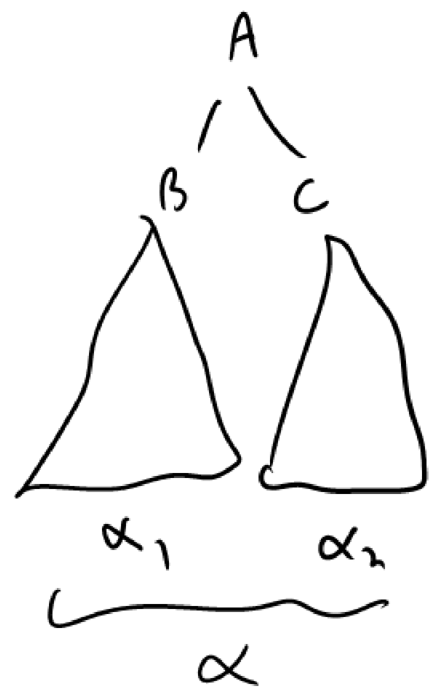
Induction. $A \Rightarrow^r \alpha \qquad r \geq 1$

$A \to BC \Rightarrow^{r-1} \alpha$ where "$A \to BC$" $\in P$.

$B \Rightarrow^s \alpha_1, \quad C \Rightarrow^t \alpha_2 \qquad \alpha = \alpha_1 \alpha_2 \quad ; s+t = r-1$

$\#NT(\alpha_1) + 2T(\alpha_1) = s+1 \quad ; \quad \#NT(\alpha_2) + 2T\#(\alpha_2) = t+1$

$\#NT(\alpha) + 2T\#(\alpha) = s+t+2 = r+1$


A
B
C
α_1
α_2
α

If $\alpha = w \in \Sigma^*$,

$\#NT(\alpha) = 0$.

$S \Rightarrow^r w$ then $2|w| = r + 1$

$r = 2|w| - 1$

Membership problem for CFGs is decidable.

Input G, w

Convert G to G' in CNF.

Enumerate all possible derivations of length $2n-1$ (where $n = |w|$)

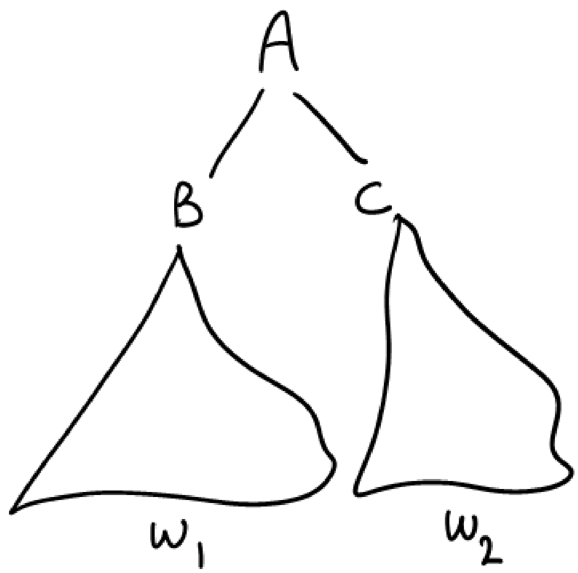
Check if any of them generates w.

If w is generated then $w \in L(G)$

and if not, $w \notin L(G)$.

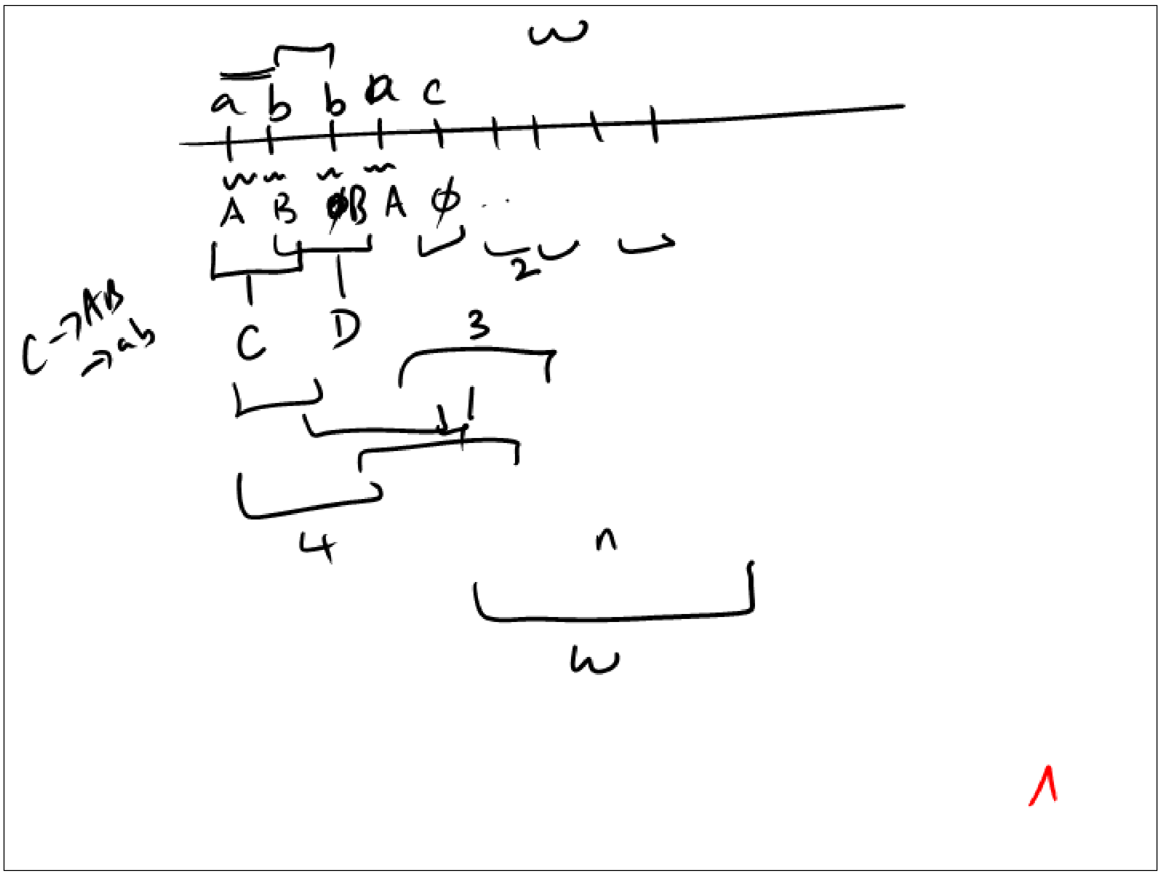
Even if G is fixed, this algorithm takes $O(\exp(n))$-time.

CYK algorithm



w

$A \Rightarrow BC$
$\Rightarrow w$



w
a b b a c
A B B A ϕ
2
C → AB
→ ab
C D
3
4
n
w

j

$$\underbrace{\underbrace{\qquad\qquad}_{\alpha} \mid \underbrace{\qquad\qquad}_{\beta}}_{D,\,E,\,A}$$

α: A, B

β: C, F, H

$D \to AF$

D, E, A

$V_{ij} = \{X \in V \mid X \Rightarrow x_{ij}\}$

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w $\quad |w| = n$

x_{ij} = substring of w starting at position i and is of length j.

$O(n^2)$ substrings of w.

B

$S \to AB \mid BC$

$A \to BA \mid a$

$B \to CC \mid b$

$C \to AB \mid a$

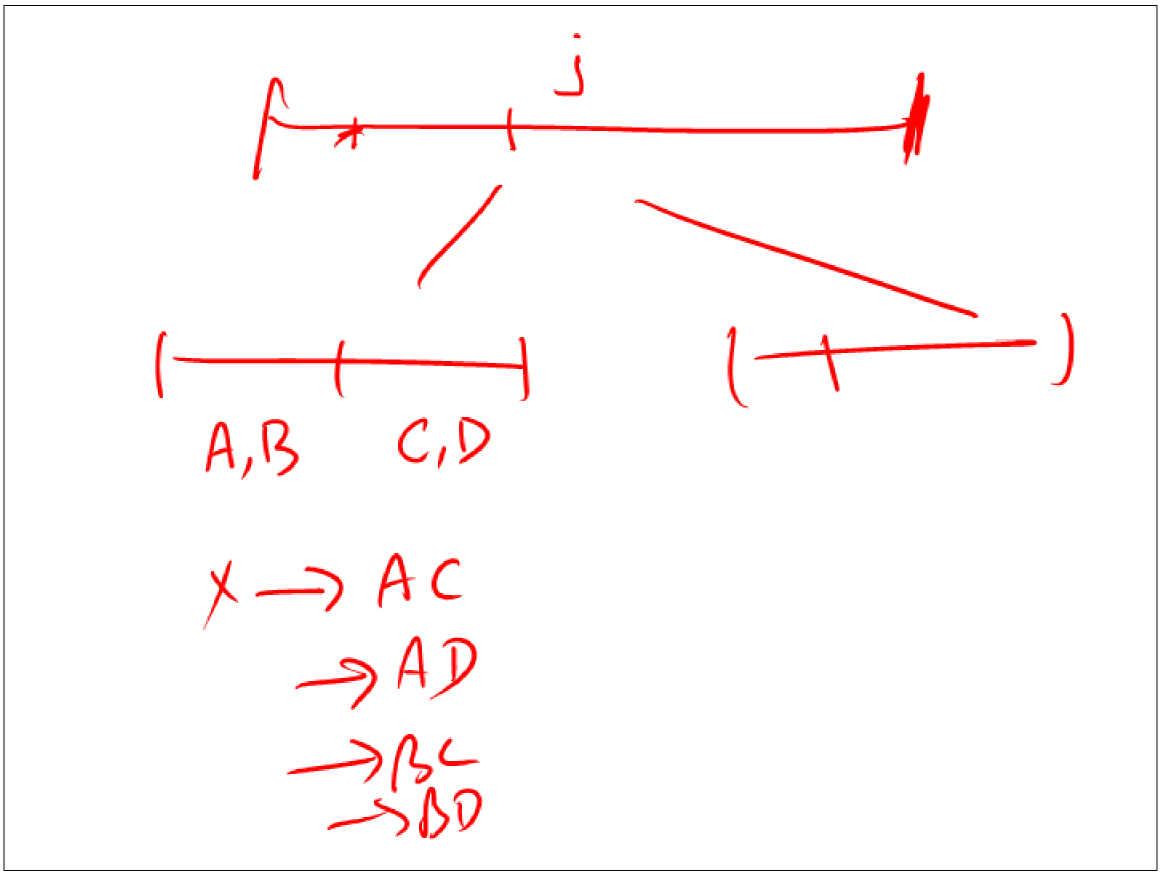
baaba $\in L(G)$

$j \downarrow$ / $i \to$	b (1)	a (2)	a (3)	b (4)	a (5)
1	B	A,C	A,C	B	A,C
2	A,S	B	C,S	A,S	
3	$\emptyset$	B	B		
4	$\emptyset$	S,C,A			
5	S,C,A				

S

Takes $O(n^3)$ steps.

$S \Rightarrow^* $ baaba, So baaba $\in L(G)$.


j
A,B
C,D
X → AC
→ AD
→ BC
→ BD

Membership problem for a fixed CFG is decidable in time $O(n^3)$ where input word w has length n.

CFLs $\subseteq$ Decidable

$L(M) = L$

CFL L $\qquad$ TM M

$\downarrow$

$G \longrightarrow \boxed{\text{Membership algorithm deciding membership in } L(G)}$

$\text{Regular Lang}_\Sigma \subseteq \text{CFL}_\Sigma$.

$V = \{ V_{q_0} , V_{q_1} \}$

Start nonterminal $= V_{q_0}$

$$V_{q_0} \rightarrow a V_{q_1} \mid b V_{q_0}$$
$$V_{q_1} \rightarrow a V_{q_0} \mid b V_{q_1}$$
$$V_{q_1} \rightarrow \epsilon$$

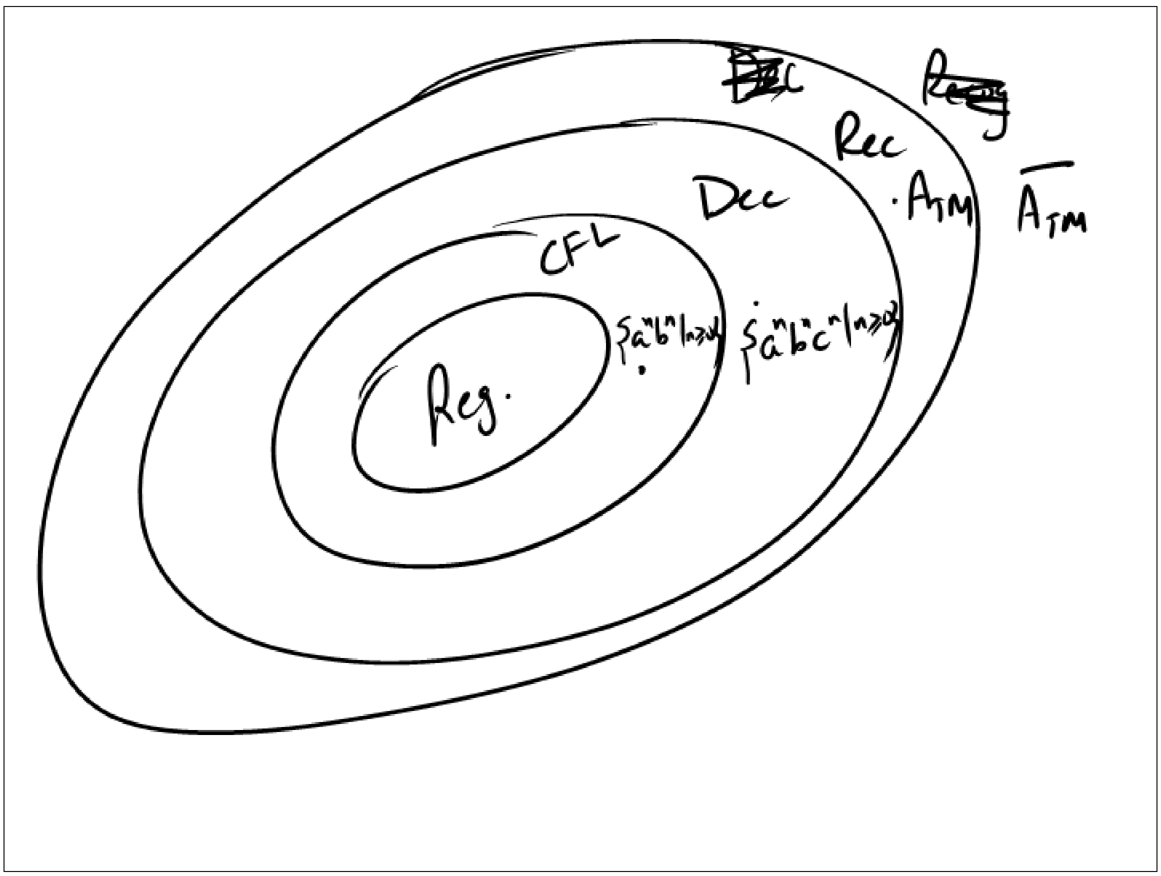
DFA $(Q, \Sigma, \delta, q_0, F)$

G (V, Σ, P, S)

$V = \{V_q \mid q \in Q\}$

$S = V_{q_0}$

$$P : \begin{cases} V_q \to a V_{q'} & \text{if } \delta(q,a) = q'. \\ V_q \to \epsilon & \text{if } q \in F. \end{cases}$$


Rec
Dec
CFL
Reg.
$\{a^n b^n \mid n \geq 0\}$
$\{a^n b^n c^n \mid n \geq 0\}$
A_{TM}
$\overline{A_{TM}}$

CFLs are closed under intersection with a regular language.

$$\underset{\text{CFL}}{L}, \ \underset{\text{Reg.}}{R} \Rightarrow \underset{\text{CFL}}{L \cap R}$$

Assume a CFG for L in CNF

Fix a DFA for R

$$A \to B\,C \quad \Big| \quad A_{\underline{q},\underline{q'}}$$

$$A_{q,q'} \to B_{q,q''}\, C_{q'',q'}$$

CFG $G = (V, \Sigma, P, S)$

$A = (Q, \Sigma, \delta, q_0, F)$

$G' = ((V \times Q \times Q) \cup \{S'\}, \Sigma, P', S')$

$S' \longrightarrow S_{q_0 q_f}$ $\quad q_f \in F$.

For every rule $A \longrightarrow BC$ in P

create the rules $A_{qq'} \longrightarrow B_{qq''} C_{q''q'}$

$\forall q, q', \forall q'' \in Q$

For every rule $A \longrightarrow a$

create rule $A_{q,q'} \longrightarrow a$ provided $\delta(q,a) = q'$.

CFG G $\cap$ $L = \{w\}$

in CNF

$$\to \bigcirc \xrightarrow{a_1} \bigcirc \xrightarrow{a_2} \xrightarrow{a_3} \cdots \xrightarrow{a_n} \circledcirc$$

So membership problem reduces to emptiness.