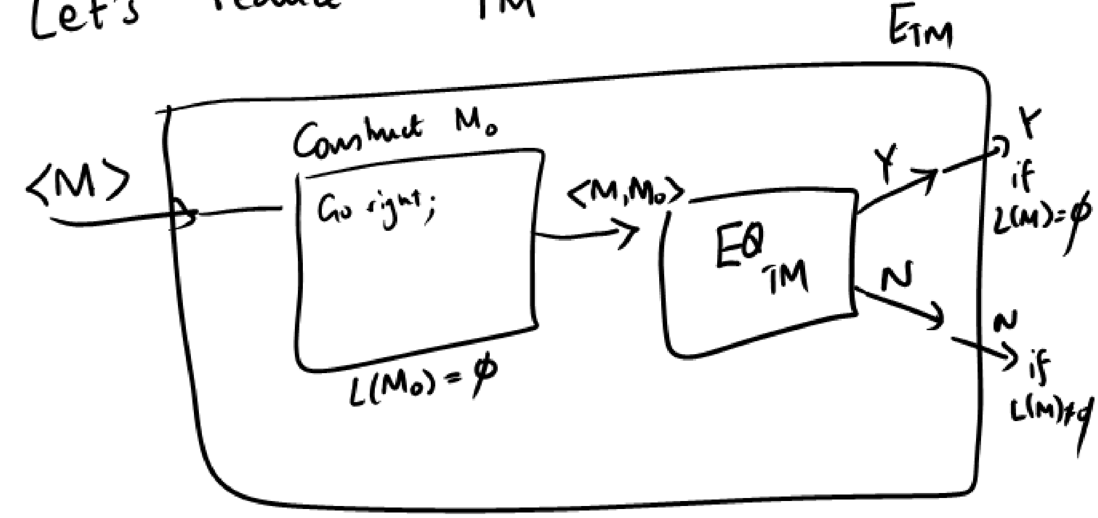
Rice's theorem

"Nothing about a TMs language is decidable"

$EQ_{TM} = \{\langle M_1, M_2\rangle \mid L(M_1) = L(M_2)\}$

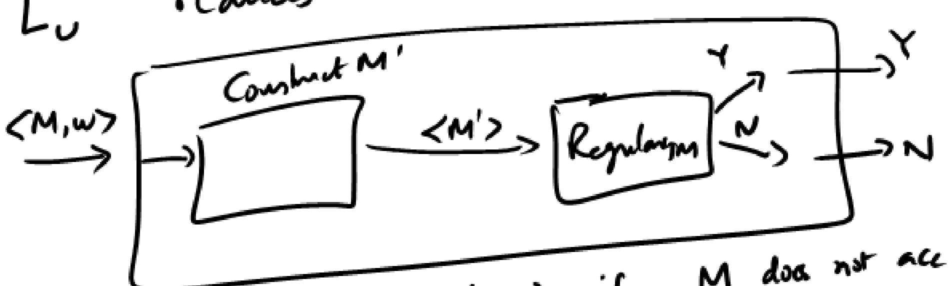
Let's reduce E_{TM} to EQ_{TM}



$Regular_{TM} = \{ \langle M \rangle \mid L(M) \text{ is regular} \}$

$L_U = \{ \langle M, w \rangle \mid M \text{ acc } w \}$

L_U reduces to $Regular_{TM}$.



$$L(M') = \begin{cases} \{0^n 1^n \mid n \in \mathbb{N}\} & \text{if } M \text{ does not acc } w \\ \Sigma^* & \text{if } M \text{ acc } w \end{cases}$$

M' [Input x
If $x = 0^n 1^n$ for some n, accept x.
else {Simulate M on w; accept x if M acc w}]

$L = \{ \langle M \rangle \mid L(M) \text{ is } \dots \}$ is undecidable. $\boxed{\text{Rice's thm}}$

s.t. L is non-trivial

i.e. $L \neq \{ \langle M \rangle \mid M \text{ is a TM} \}$

nor $L = \{ \langle M \rangle \mid M \text{ is a TM} \}$.

Rice's theorem.

Let L be a language of TMs.

Also let

a) Membership of $\langle M \rangle$ in L depends only on $L(M)$.

Formally if $\langle M_1 \rangle$ and $\langle M_2 \rangle$ are TMs, with $L(M_1) = L(M_2)$,

~~then eit~~

then either $(\langle M_1 \rangle \in L \wedge \langle M_2 \rangle \in L)$

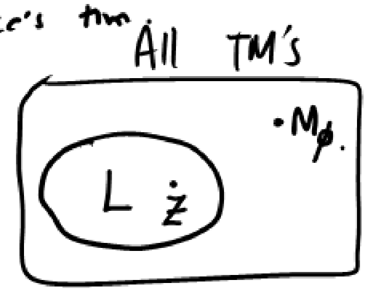
or $(\langle M_1 \rangle \notin L \wedge \langle M_2 \rangle \notin L)$

b) L is non-trivial, ie. $L \neq \emptyset$ and $L \neq \{\langle M \rangle \mid M \text{ is a TM}\}$

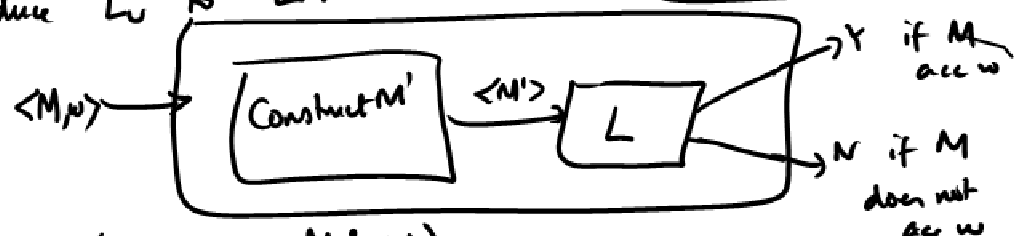
Then L is undecidable.

Pf. Let L be as in Rice's thm.

Let M_ϕ be some TM accepting ϕ.



Reduce L_u to L.



Code for M' (depends on M & w):

Input is x

1) Simulate M on w
2) If M rejects w, halt and reject
3) If M accepts w, simulate Z on x, and accept iff Z accepts.

If M acc w, $L(M') = L(Z) \neq \phi$

If M does not acc w, $L(M') = \phi = L(M_\phi)$

If $M_\emptyset \in L$, consider

$$L' = \{\langle M \rangle \mid M \text{ is a TM}\} \setminus L$$

Follow proof to show L' is not decidable.

Since decidable languages are closed under complement

L is also undecidable.