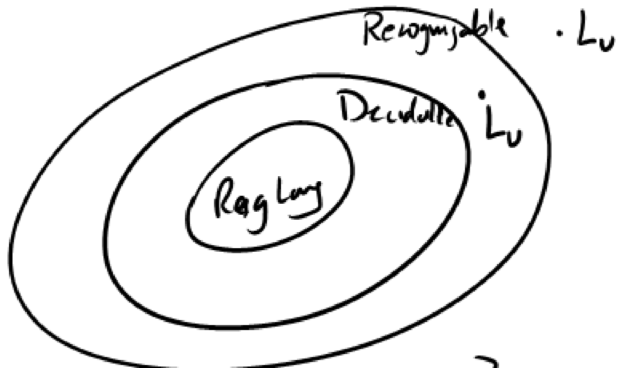
Lecture # 16:

Undecidable Languages/

Unsolvable problems

This statement is false.

Gödel "This statement is not provable"



$$L_U = \{ \langle M, w \rangle \mid M \text{ accepts } w \}$$

L_U is recognizable because
a TM ~~that~~ can simulate M on w,
and accept if M accepts w.

This TM (UTM) may not halt on all inputs.

Diagonalization

$\Sigma = \{0,1\}$

w is an infinite string if $w = a_0 a_1 \dots$, $a_i \in \Sigma$

	1	2	3	4	5 …
w_1	a^1_ϕ	a^1_2	a^1_3	a^1_4	…
w_2	a^2_1	a^2_2	a^2_3	…	…
w_3					
$\vdots$					

$\overline{a^1_1}\,\overline{a^2_2}\,\overline{a^3_3} \dots \overline{a^i_i} \dots$

$d = w_j$

$d[j] = \overline{w_j[j]}$

L_U is not TM-decidable.
D is not TM-recognizable.

Enumerate all TMs.

For any Σ, Σ^* is countable and we can Σ^* in a computable manner.

0-length : ϵ
1-length : a, b
2-length : aa, ab, ba, bb
3 : -
⋮

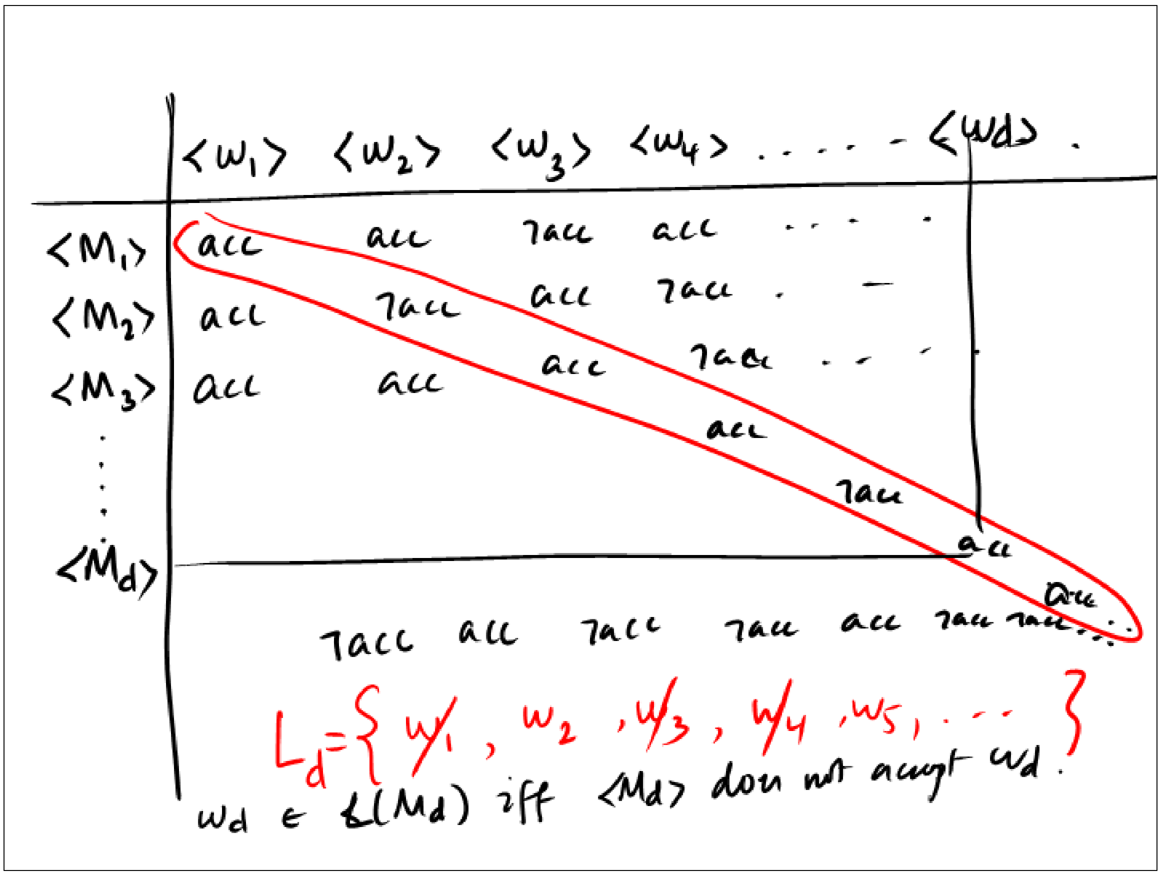
ϵ, a, b, aa, ab, ba, bb, aaa, ...

The ordering is computable:

TM: Input i
Generate words in the order
Outputs w_i.

The inverse is computable.

TM: Input w.
Generate words in the order till you hit w.
Output i s.t. $w_i = w$


<w1> <w2> <w3> <w4> . . . . . <wd> .
<M1> acc acc ¬acc acc . . . .
<M2> acc ¬acc acc ¬acc . —
<M3> acc acc acc ¬acc . . .
acc
¬acc
<Md> acc
acc
¬acc acc ¬acc ¬acc acc ¬acc ¬acc . . .
Ld = { w1 , w2 , w3 , w4 , w5 , . . . }
wd ∈ L(Md) iff <Md> does not accept wd .

$$L_d = \{ w_i \mid M_i \text{ does not accept } w_i \}$$

is not TM-recognizable.

Proof. If L_d is TM-recognizable, there is a TM M s.t. $L(M) = L_d$.

Now $M = M_i$ for some i.

Now we consider w_i

~~$w \in L_d$~~ $w_i \in L_d$ iff M_i does not accept w_i

So $L_d \neq L(M_i)$

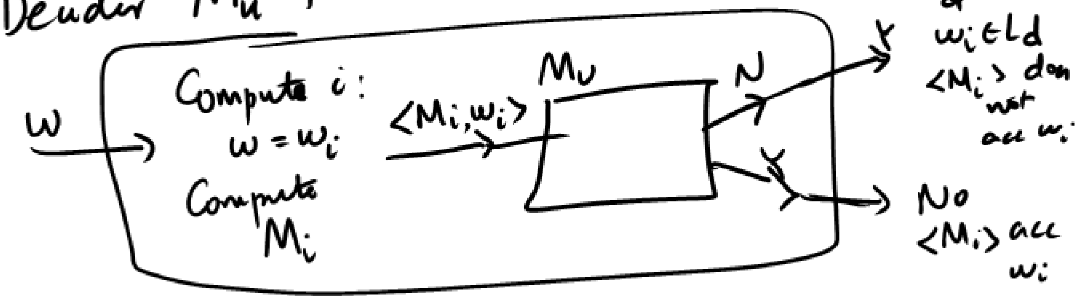
So now we know

$L_d = \{ w_i \mid \langle M_i \rangle$ does not accept $\langle w_i \rangle \}$ is not TM-recognizable

~~E~~ Hence L_d is not decidable.

We show $L_U = \{ \langle M, w \rangle \mid M$ accepts $w \}$ is not decidable.

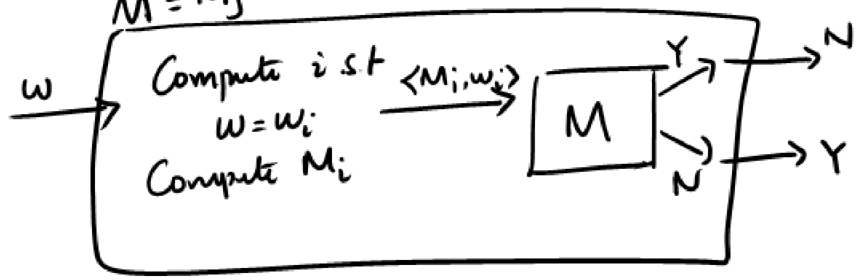
Assume Decider M_u for L_U exists.   Decider for L_d



$L_U = \{ \langle M, w \rangle \mid M \text{ acc } w \}$ is ~~undecidable~~ undecidable

Assume L_U is decidable. Let M be a decider for L_U.

$\hat{M} = M_j$



$\hat{M} = M_j$ for some j.

Does $\hat{M}$ accept w_j?

If $\hat{M}$ acc w_j, then M must have said "No"

i.e. M_j does not acc w_j

$\hat{M}$ does not acc w_j

If $\hat{M}$ does not acc w_j; i.e. then M must say "yes", $\hat{M}$ acc w_j

$L = \{ \langle \underline{C}, \underline{w} \rangle \mid C$ is a C-pgm that accepts $w \}$

There is no C-pgm deciding L.

$L(\hat{M}) = \{ w_i \mid M_i$ does not accept $w_i \}$

$w_j \in L(\hat{M})$ or not?

$w_j \in L(M_j)$ iff M_j does not accept w_j

iff $w_j \in L(M_j)$

All TMs is effectively enumerable.

$TM_1, TM_2, TM_3 \ldots\ldots$

All ~~inputs~~ words in Σ^* are enumerable

M

$L = \{ w \mid M \text{ does not acc } w \}$

$\overline{L(M)}$