Lecture #10

Proving languages non-regular

- Using MNT
- Using pumping lemma.

<u>MNT</u>.

L is regular iff the number of suffix languages of L is finite (i.e. $\mathcal{C}(L)$ is finite).

If $|\mathcal{C}(L)| = k$ then the minimal DFA for L has k states and is unique (there is a DFA for L with k states & there is no DFA for L with <k states & there is only one DFA for L with k states).

If $\mathcal{C}(L)$ is infinite $\Rightarrow$ L is not regular

How do we show that L is not regular?

Exhibit infinitely many distinct suffix languages

$[\![L/x_1]\!], [\![L/x_2]\!], \ldots\ldots$

We must exhibit an infinite set

$S = \{x_1, x_2, \ldots\}$

s.t. $\forall x, y \in S$ s.t. $x \neq y$, $[\![L/x]\!] \neq [\![L/y]\!]$

Or $\forall x, y \in S$ s.t. $x \neq y$, $\exists z. ((z \in [\![L/x]\!] \wedge z \notin [\![L/y]\!])$
$\vee (z \notin [\![L/x]\!] \wedge z \in [\![L/y]\!])$

$\forall x, y \in S$ s.t $x \neq y$. $\exists z. ((xz \in L \wedge yz \notin L) \vee (xz \notin L \wedge yz \in L))$

Technique for proving non-regularity (MNT)

Let $L \subseteq \Sigma^*$.

If there is an infinite set $S \subseteq \Sigma^*$ s.t.

$\forall x, y \in S$ with $x \neq y$,

$$\exists z . \big((xz \in L \land yz \notin L) \lor (xz \notin L \land yz \in L) \big)$$

then L is not a regular language.

Eg. $L = \{ a^n b^n \mid n \in \mathbb{N} \}$

Choose $S = \{ a^i \mid i \in \mathbb{N} \} = aa^*$

S is infinite. Let $x = a^i$ and $y = a^j$ with $i \neq j$.

Choose $z = b^i$. Then $xz \in L$ and $yz = a^j b^i \notin L$.

Hence L is not regular.

Eg. Let us show $L = \{w \in \{0,1\}^* \mid w$ has an ~~equ~~ equal number of 0's and 1's$\}$.

is not regular.

Proof. (MNT technique)

$S = \{0^i \mid i \in \mathbb{N}\} = 00^*$

S is infinite. Let $x, y \in S$, $x \neq y$, $x = 0^i$, $y = 0^j$, $i \neq j$.

Then chose $z = 1^i$.

Then $xz \in L$ and $yz \notin L$.

So L is not regular.

Eg. $L = \{ 1^{n^2} \mid n \in \mathbb{N} \}$ is not regular.

$$\underbrace{1, 4}_{3}, \underbrace{4, 9}_{5}, \underbrace{9, 16}_{7}, \underbrace{16, 25}_{9}, 36, \ldots \underbrace{n^2 . (n+1)^2}_{2n+1}$$

$S = L$

S is infinite.
Let $x, y \in S$, $x \neq y$. So let $x = 1^{i^2}$ $y = 1^{j^2}$ $(i \neq j)$.

Wlog $i < j$

Choose $z = 1^{2i+1}$.

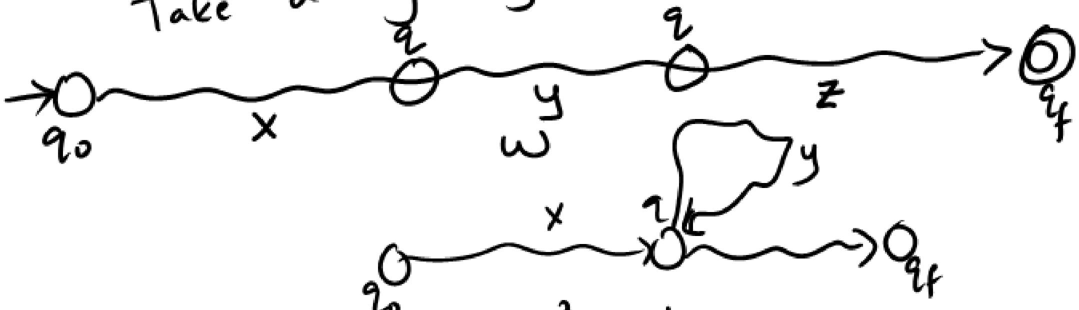
$xz = 1^{i^2} 1^{2i+1} = 1^{(i+1)^2} \in L$

$yz = 1^{j^2} 1^{2i+1} = 1^{j^2+2i+1} \notin L$

the smallest r s.t. j^2+r is a square is $2j+1$
$2i+1 < 2j+1$

Pumping Lemma

Let L be regular.
Then there is a DFA for it, say with n states.
Take a very long word in L, say w



So $xy^2z \in L$ $\quad xy^3z \in L$ ...

$xz \in L$

$\forall i \in \mathbb{N}_0 . \ xy^iz \in L.$

$y \neq \epsilon$
$|xy| \leq n.$

$\alpha \Rightarrow \beta$

$\neg\beta \Rightarrow \neg\alpha$

Pumping Lemma.

If L is regular then

$\exists p \in \mathbb{N}.\ \forall w \in L$. s.t. $|w| > p$,

$\exists x, y, z \in \Sigma^* . (w = xyz$

$\wedge |y| > 0$

$\wedge |xy| \leq p$

$\wedge \forall i \in \mathbb{N}_0 . xy^iz \in L)$.

$\forall p \in \mathbb{N}.\ \exists w \in L\ |w| > p|,$

$\forall x, y, z \in \Sigma^*$ with $w = xyz \wedge |y| > 0 \wedge |xy| \leq p$,

$\exists i \in \mathbb{N}_0\ xy^iz \notin L$.

$\Rightarrow L$ is not regular.

Eg. $L = \{a^n b^n \mid n \in \mathbb{N}\}$

Let $p \in \mathbb{N}$ be an arbitrary number.

Choose $w = a^p b^p$ $\quad w \in L \quad |w| > p$

Let $w = xyz$ for an arbitrary $x, y, z \in \Sigma^*$,

with $|y| > 0$, $|xy| \le p$

Since $|xy| < p$, $xy \in a^*$.

$y \neq \epsilon$, $y = a^j$ for some $j > 0$.

Choose $i = 2$.

Then $xy^2z = a^p a^j b^p \quad (j > 0)$

$\notin L$.

Hence L is not regular.

$L = \{ ww \mid w \in \Sigma^* \}$ is not regular $\quad \Sigma = \{a, b\}$

Proof (PL) Let $p \in \mathbb{N}$ be an arbitrary number.

$w = a^p a^p$ X will not work

$w = (ab)^{2p}$ X

$w = a^p b a^p b$

Let $x, y, z \in \Sigma^*$. $w = xyz$, $|xy| \leq p$, $|y| > 0$.

$xy \in a^*$ $\quad y = a^j \quad j > 0$

Choose $i = 2$ $\quad xy^2z = a^{p+j} b a^p b \notin L$.

So L is not regular.