Reductions

X reduces to Y

if I can decide X using a decider for Y.

Means that X is only as hard as Y

If Y is decidable, then X is decidable.

If X is undecidable, then Y is undecidable

Y harder

↑

X easier

Suppose X is undecidable
and we want to show Y is undecidable

It suffices to show that
X reduces to Y.

Y
$\uparrow$
X

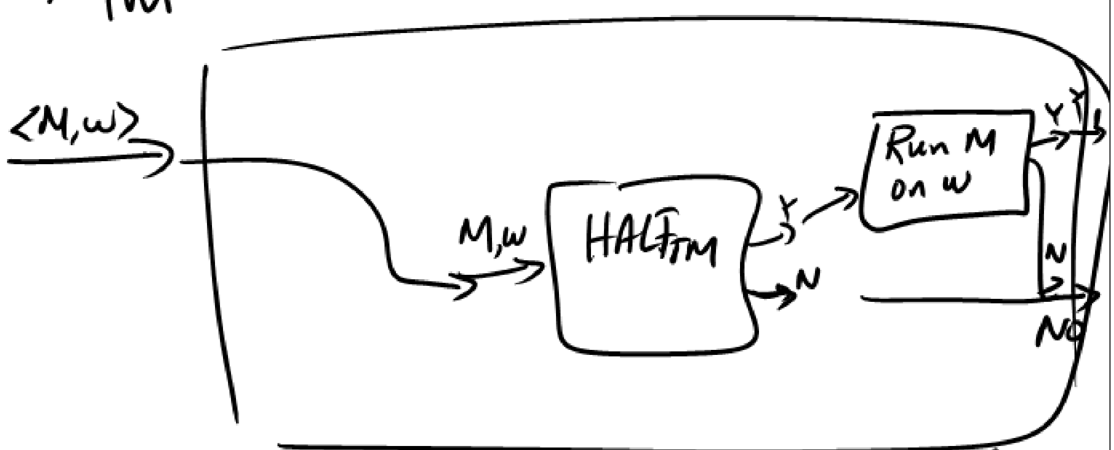
i.e. We can decide X using a
decider for Y.

Then Y is undecidable:

[Proof If Y was decidable, X would
be decidable, which is not true.
So Y is undecidable]

$HALT_{TM} = \{ \langle M, w \rangle \mid M \text{ halts on } w \}$

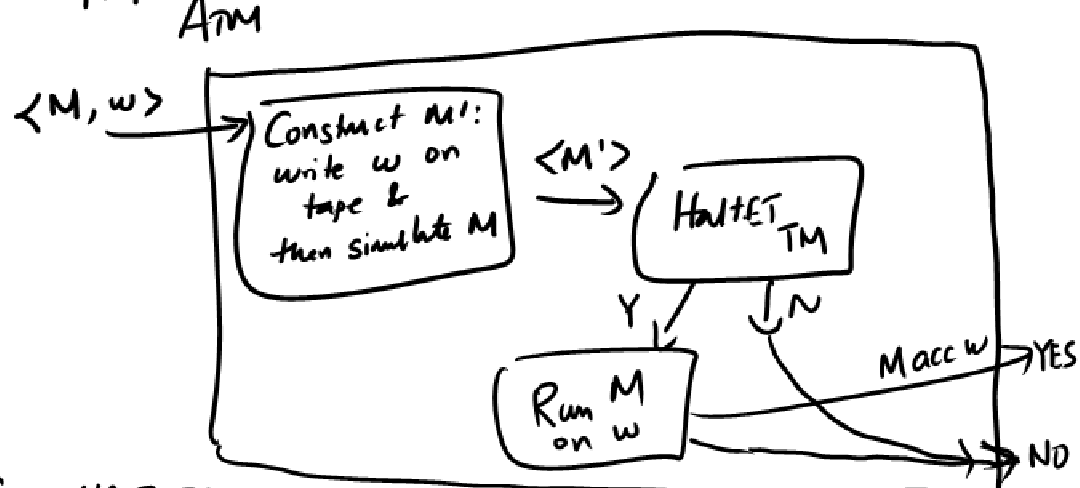
A_{TM} reduces to $HALT_{TM}$



So $HALT_{TM}$ is undecidable (since A_{TM} is undec.)

$HALTET_{TM} = \{ \langle M \rangle \mid M$ is a TM that halts when started on the empty tape $\}$

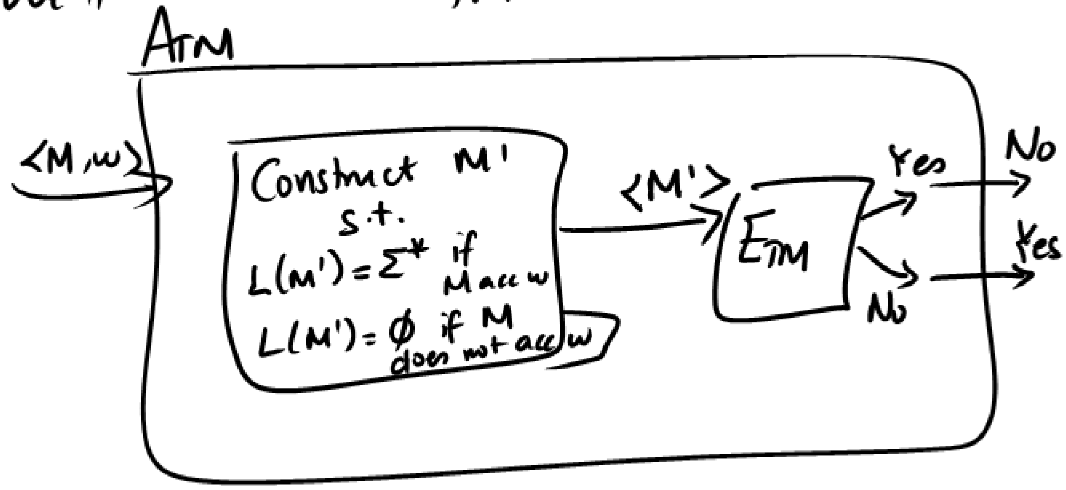
A_{TM} reduces to $HALT\text{-}ET_{TM}$



So $HALT\text{-}ET_{TM}$ is undecidable.

$$E_{TM} = \{\langle M\rangle \mid \underset{M \text{ is a TM}}{L(M) = \emptyset}\}.$$

We'll show A_{TM} reduces to E_{TM}.



So E_{TM} is undecidable

Construction of M':

Given $\langle M, w \rangle$,

construct $M'_{\langle M,w \rangle}$ which does the following:

- M' gets as input x
- M' ignore x, and run M on w.
- If M accepts w, accept x.

If M accepts w then $L(M') = \Sigma^*$.

If M does not accept w then $L(M') = \phi$

$\text{Regular}_{TM} = \{ \langle M \rangle \mid \mathcal{L}(M) \text{ is regular, } M \text{ is a TM} \}$

A_{TM} reduces to Regular_{TM}

A_{TM} gets as input $\langle M, w \rangle$.

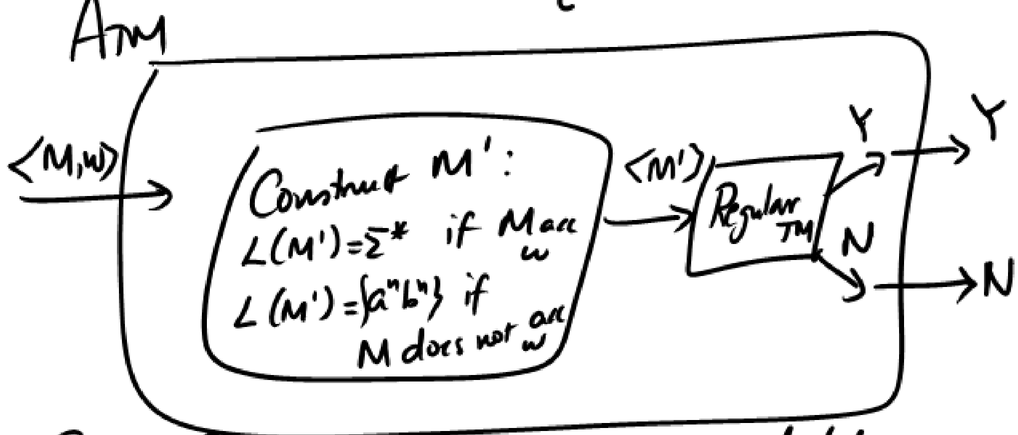
Construct M' (based on M, w) as:

- M' gets as input x
- If $x = a^n b^n$ for some n, halt and accept.
- If not, simulate M on w.
- If M halts and accepts w, accept x.

If M accepts w, $L(M') = \Sigma^*$

If M does not accept w,

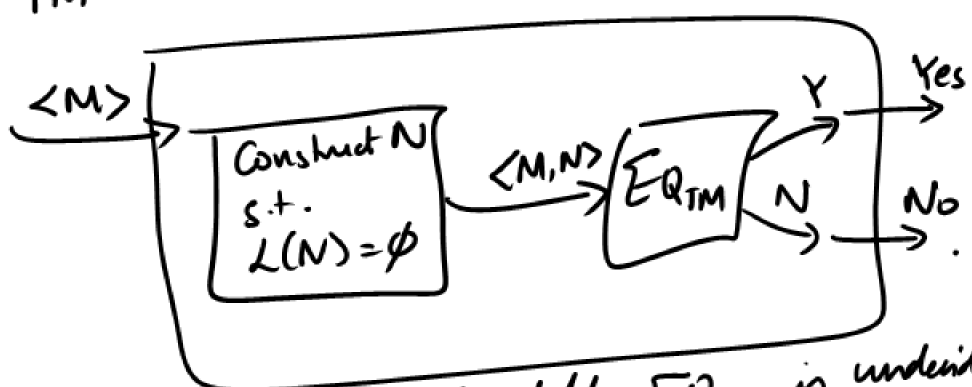
$$L(M') = \{ a^n b^n \mid n \in \mathbb{N} \}.$$



So $Regular_{TM}$ is undecidable.

$EQ_{TM} = \{ \langle M, N \rangle \mid M$ and N are TMs. and $L(M) = L(N) \}$

E_{TM} reduces to EQ_{TM}.



Since E_{TM} is undecidable, EQ_{TM} is undecidable $\square$