1 Unrestricted Computation

General Computing Machines

- Machines so far: DFAs, NFAs, PDAs
  - Limitations on how much memory they can use: fixed amount of memory plus (for PDAs) a stack
  - Limitations on what they can compute/decide: only regular languages or context free languages
- The complete machine?
  - No limitations on memory usage? And maybe other ways to use computational resources that we haven't thought of...
    - Come up with a model that describes all "conceivable" computation
  - No limitation on what they can compute?
    - No! There are far too many languages over $\{0, 1\}$ than there are "machines" or programs (as long as machines can be represented digitally)

General Computing Machines

Alonzo Church, Emil Post, and Alan Turing (1936)

Figure 1: Alonzo Church

Figure 2: Emil Post

Figure 3: Alan Turing

- Church (λ-calculus), Post (Post's machine), Turing (Turing machine) independently came up with formal definitions of mechanical computation
- All equivalent!
- In this course: Turing Machines

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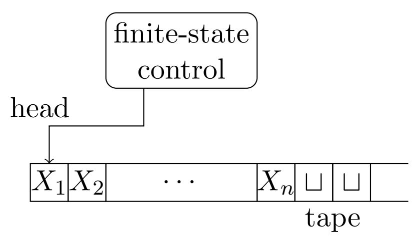
2 Turing Machines

2.1 Definition

Turing Machines



- Unrestricted memory: an infinite tape
  - A finite state machine that reads/writes symbols on the tape
  - Can read/write anywhere on the tape
  - Tape is infinite in one direction only (other variants possible)
- Initially, tape has input and the machine is reading (i.e., tape head is on) the leftmost input symbol.
- Transition (based on current state and symbol under head):
  - Change control state
  - Overwrite a new symbol on the tape cell under the head

– Move the head left, or right.

Turing Machines

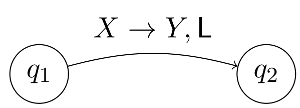
Formal Definition

A Turing machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathsf{acc}}, q_{\mathsf{rej}})$ where

- Q is a finite set of control states
- Σ is a finite set of input symbols
- $\Gamma \supseteq \Sigma$ is a finite set of tape symbols. Also, a blank symbol $\sqcup \in \Gamma \setminus \Sigma$
- $q_0 \in Q$ is the initial state
- $q_{\mathsf{acc}} \in Q$ is the accept state
- $q_{\mathsf{rej}} \in Q$ is the reject state, where $q_{\mathsf{rej}} \neq q_{\mathsf{acc}}$
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$ is the transition function.

  Given the current state and symbol being read, the transition function describes the next state, symbol to be written and direction (left or right) in which to move the tape head.

Transition Function



$\delta(q_1, X) = (q_2, Y, \mathsf{L})$: Read transition as "the machine when in state q_1, and reading symbol X under the tape head, will move to state q_2, overwrite X with Y, and move its tape head to the left"

- In fact $\delta : (Q \setminus \{q_{\mathsf{acc}}, q_{\mathsf{rej}}\}) \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$. No transition defined after reaching q_{acc} or q_{rej}
- Transitions are deterministic
- Convention: if $\delta(q, X)$ is not explicitly specified, it is taken as leading to q_{rej}, i.e., say $\delta(q, X) = (q_{\mathsf{rej}}, \sqcup, \mathsf{R})$

Configurations

The configuration (or "instantaneous description") contains all the information to exactly capture the "current state of the computation"

$$X_1 X_2 \cdots X_{i-1} q X_i \cdots X_n$$

- Includes the current state: q
- Position of the tape head: Scanning i^{th} symbol X_i
- Contents of all the tape cells till the rightmost nonblank symbol. This is will always be finitely many cells. Those symbols are $X_1X_2\cdots X_n$, where $X_n \neq \sqcup$ unless the tape head is on it.

Special Configurations

- Start configuration: $q_0X_1\cdots X_n$, where the input is $X_1\cdots X_n$
- Accept and reject configurations: The state q is q_{acc} or q_{rej}, respectively. These configurations are *halting configurations*, because there are no transitions possible from them.

Single Step

Definition 1. We say one configuration (C_1) *yields* another (C_2), denoted as $\mathrm{C}_1 \vdash \mathrm{C}_2$, if one of the following holds.

- If $\delta(q, X_i) = (p, Y, \mathsf{L})$ then

$$X_1X_2\cdots X_{i-1}qX_iX_{i+1}\cdots X_n \vdash X_1X_2\cdots X_{i-2}pX_{i-1}YX_{i+1}\cdots X_n$$

  Boundary Cases:

  - If $i = 1$ then $qX_1X_2\cdots X_n \vdash pYX_2\cdots X_n$
  - If $i = n$ and $Y = \sqcup$ then $X_1\cdots X_{n-1}qX_n \vdash X_1\cdots pX_{n-1}$

- If $\delta(q, X_i) = (p, Y, \mathsf{R})$ then

$$X_1X_2\cdots X_{i-1}qX_iX_{i+1}\cdots X_n \vdash X_1X_2\cdots X_{i-1}YpX_{i+1}\cdots X_n$$

  Boundary Case:

  - If $i = n$ then $X_1\cdots X_{n-1}qX_n \vdash X_1\cdots X_{n-1}Yp\sqcup$

Computations

Definition 2. We say $\mathrm{C}_1 \vdash^* \mathrm{C}_2$ if the machine can move from C_1 to C_2 in zero or more steps, i.e., $\mathrm{C}_1 = \mathrm{C}_2$ or there exist $\mathrm{C}'_1, \ldots, \mathrm{C}'_n$ such that $\mathrm{C}_1 = \mathrm{C}'_1$, $\mathrm{C}_2 = \mathrm{C}'_n$ and $\mathrm{C}'_i \vdash \mathrm{C}'_{i+1}$

Acceptance and Recognition

Definition 3. A Turing machine M *accepts* w iff $q_0 w \vdash^* \alpha_1 q_{\mathsf{acc}} \alpha_2$, where α_1, α_2 are some strings. In other words, the machine M when started in its intial state and with w as input, reaches the accept state.

Note: The machine may not read all the symbols in w. It may pass back and forth over some symbols of w several times. Finally, w may have been completely overwritten.

Definition 4. For a Turing machine M, define $\mathbf{L}(M) = \{w \mid M \text{ accepts } w\}$. M is said to accept or *recognize* a language L if $L = \mathbf{L}(M)$.

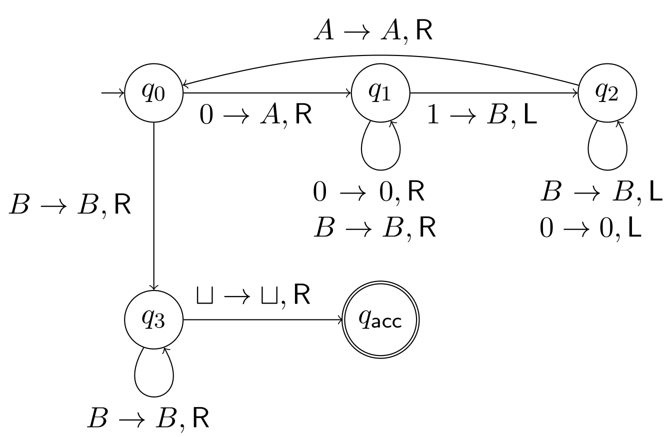
2.2 Examples

Example 1: TM for $\{0^n 1^n \mid n > 0\}$
Design a TM to accept the language $L_{0n1n} = \{0^n 1^n \mid n > 0\}$

High level description

```
On input string w
    while there are unmarked 0s, do
        Mark the left most 0
        Scan right till the leftmost unmarked 1;
            if there is no such 1 then crash
        Mark the leftmost 1
    done
    Check to see that there are no unmarked 1s;
        if there are then crash
    accept
```

Example 1: TM for $\{0^n 1^n \mid n > 0\}$



- Accepts input 0011:

$$q_0 0011 \vdash A q_1 011 \vdash A0 q_1 11 \vdash A q_2 0B1 \vdash q_2 A0B1 \vdash A q_0 0B1 \vdash AA q_1 B1 \vdash AAB q_1 1 \vdash AA q_2 BB \vdash A q_2 ABB \vdash A$$

- Rejects input 00:

$$q_0 00 \vdash A q_1 0 \vdash A0 q_1 \sqcup \vdash A0 \sqcup q_{\mathsf{rej}} \sqcup$$

Example: $\{0^n 1^n \mid n > 0\}$
Formal Definition

The machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathsf{acc}}, q_{\mathsf{rej}})$ where

- $Q = \{q_0, q_1, q_2, q_3, q_{\mathsf{acc}}, q_{\mathsf{rej}}\}$
- $\Sigma = \{0, 1\}$, and $\Gamma = \{0, 1, A, B, \sqcup\}$
- δ is given as follows

$$\begin{array}{ll}
\delta(q_0, 0) = (q_1, A, R) & \delta(q_0, B) = (q_3, B, R) \\
\delta(q_1, 0) = (q_1, 0, R) & \delta(q_1, B) = (q_1, B, R) \\
\delta(q_1, 1) = (q_2, B, L) & \delta(q_2, B) = (q_2, B, L) \\
\delta(q_2, 0) = (q_2, 0, L) & \delta(q_2, A) = (q_0, A, R) \\
\delta(q_3, B) = (q_3, B, R) & \delta(q_3, \sqcup) = (q_{\mathsf{acc}}, \sqcup, R)
\end{array}$$

In all other cases, $\delta(q, X) = (q_{\mathsf{rej}}, \sqcup, R)$. So for example, $\delta(q_0, 1) = (q_{\mathsf{rej}}, \sqcup, R)$.

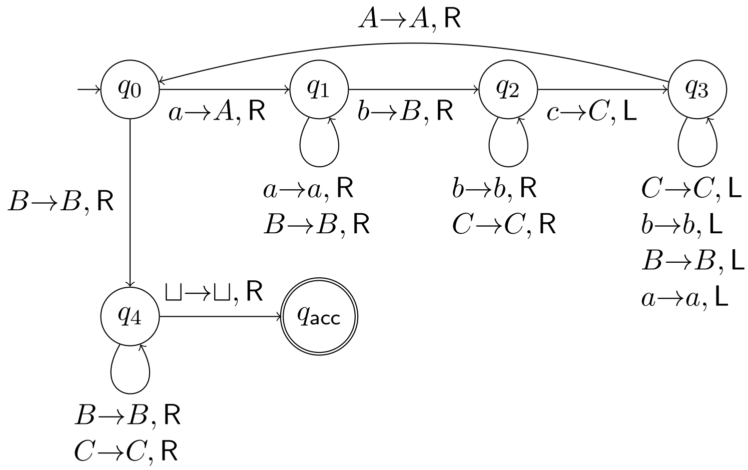
Example 2: TM for $\{a^n b^n c^n \mid n > 0\}$

Design a TM to accept the language $L_{anbncn} = \{a^n b^n c^n \mid n > 0\}$

High level description

```
On input string w
    while there are unmarked as, do
        Mark the left most a
        Scan right to reach the leftmost unmarked b;
            if there is no such b then crash
        Mark the leftmost b
        Scan right to reach the leftmost unmarked c;
            if there is no such c then crash
        Mark the leftmost c
    done
    Check to see that there are no unmarked cs or cs;
        if there are then crash
    accept
```

Example 2: TM for $\{a^nb^nc^n \mid n > 0\}$



$q_0aabbcc\vdash^* AaBq_3bCc\vdash^* q_3AaBbCc \vdash Aq_0aBbCc\vdash^* AAq_0BBCC\vdash^* AABBCCq_4\sqcup \vdash AABBCC\sqcup q_{\mathsf{acc}}\sqcup$

Deciding a Language

- Only halting configurations are those with state q_{acc} or q_{rej}
- A Turing machine may keep running forever on some input
- Then the machine does not accept that input
- So two ways to not accept: reject or never halt

Definition 5. A Turing machine M is said to *decide* a language L if $L = \mathbf{L}(M)$ and M halts on every input

Deciding a language is more than recognizing it. There are languages which are recognizable, but not decidable.