CS 373: Theory of Computation

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1 Unrestricted Grammars

1.1 Overview

Grammars

Definition 1. A grammar is $G = (V, \Sigma, R, S)$, where

- V is a finite set of variables/non-terminals
- Σ is a finite set of terminals
- $S \in V$ is the start symbol
- $R \subseteq (\Sigma \cup V)^* \times (\Sigma \cup V)^*$ is a finite set of rules/productions

We say $\gamma_1 \alpha \gamma_2 \Rightarrow_G \gamma_1 \beta \gamma_2$ iff $(\alpha \to \beta) \in R$. And $L(G) = \{w \in \Sigma^* \mid S \overset{*}{\Rightarrow}_G w\}$

Example

Example 2. Consider the grammar G with $\Sigma = \{a\}$ with

$$
\begin{array}{lll}
S \to \$Ca\# \mid a \mid \epsilon & Ca \to aaC & \$D \to \$C \\
C\# \to D\# \mid E & aD \to Da & aE \to Ea \\
\$E \to \epsilon &&
\end{array}
$$

The following are derivations in this grammar

$$
\begin{aligned}
S &\Rightarrow \$Ca\# \Rightarrow \$aaC\# \Rightarrow \$aaE \Rightarrow \$aEa \Rightarrow \$Eaa \Rightarrow aa \\
S &\Rightarrow \$Ca\# \Rightarrow \$aaC\# \Rightarrow \$aaD\# \Rightarrow \$aDa\# \Rightarrow \$Daa\# \Rightarrow \$Caa\# \\
&\Rightarrow \$aaCa\# \Rightarrow \$aaaaC\# \Rightarrow \$aaaaE \Rightarrow \$aaaEa \Rightarrow \$aaEaa \\
&\Rightarrow \$aEaaa \Rightarrow \$Eaaaa \Rightarrow aaaa
\end{aligned}
$$

$L(G) = \{a^i \mid i \text{ is a power of } 2\}$

Grammars for each task

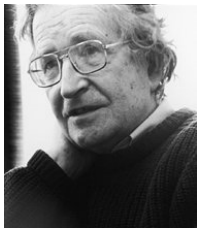

Figure 1: Noam Chomsky

- What is the expressive power of these grammars?

- Restricting the types of rules, allows one to describe different aspects of natural languages
- These grammars form a hierarchy

1.2 Expressive Power

Type 0 Grammars

Definition 3. Type 0 grammars are those where the rules are of the form

$$\alpha \to \beta$$

where $\alpha, \beta \in (\Sigma \cup V)^*$

Example 4. Consider the grammar G with $\Sigma = \{a\}$ with

$$\begin{array}{lll} S \to \$Ca\# \mid a \mid \epsilon & Ca \to aaC & \$D \to \$C \\ C\# \to D\# \mid E & aD \to Da & aE \to Ea \\ \$E \to \epsilon & & \end{array}$$

Expressive Power of Type 0 Grammars

Theorem 5. *L is recursively enumerable iff there is a type 0 grammar G such that $L = L(G)$.*

Thus, type 0 grammars are as powerful as Turing machines.

Recognizing Type 0 languages

Proposition 6. *If $G = (V, \Sigma, R, S)$ is a type 0 grammar then $L(G)$ is recursively enumerable.*

Proof. We will show that $L(G)$ is recognized by a 2-tape non-deterministic Turing machine M, with tape 1 storing the input w, and tape 2 used to construct a derivation of w from S. □

Recognizing Type 0 Grammars

Proof (contd).
- At any given time tape 2, stores the current string of the derivation; initial tape contains S.
- To simulate the next derivation step, M will (nondeterministically) choose a rule to apply, scan from left to right and choose (nondeterministically) a position to apply the rule, replace the substring matching the LHS of the rule with the RHS to get the string at the next step of derivation.

- If tape 2 contains only terminal symbols, then M will check to see if it matches tape 1. If so, the input is accepted, else it is rejected. □

Describing R.E. Languages

Proposition 7. *If L is recursively enumerable, then there is a type 0 grammar G such that $L = L(G)$.*

Proof. Let M be a Turing machine recognizing L. The grammar G will simulate M "backwards" starting from an accepting configuration.

- A string γ in the derivation will encode a configuration of M
- G has rules such that $\gamma_1 \Rightarrow \gamma_2$ iff M in one step can move from configuration γ_2 to configuration γ_1
- The rules from the start symbol will first generate some accepting configuration of M
- Once (some) initial configuration $q_0 w$ is generated, rules in G will erase symbols to produce the terminal w.

For the TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathsf{acc}}, q_{\mathsf{rej}})$ recognizing L, $G = (V, \Sigma, R, S)$ where

- $V = (\Gamma \setminus \Sigma) \cup Q \cup \{S, A, \lhd, \rhd\}$

We will now describe the rules.

First the grammar will generate an accepting configuration between the symbols $\rhd$ and $\lhd$. Recall, that an accepting configuration is a string of the form $\alpha q_{\mathsf{acc}} \beta$ where $\alpha, \beta \in \Gamma^*$. Rules to that are as follows.

$$S \to \rhd A q_{\mathsf{acc}} A \lhd \quad A \to aA \mid \epsilon$$

In the above rules a is any symbol in Γ; so we have rule $A \to aA$ for *every* symbol $a \in \Gamma$.

Next, there will be rules to simulate one step of M backwards. If $\delta(q, a) = (q', b, \mathsf{L})$ then have a rule $q'cb \to cqa$; this is because if the configuration is of the form $\alpha cqa\beta$ then taking a step according to this transition will result in the configuration $\alpha q'cb\beta$. For similar reasons, if $\delta(q, a) = (q', b, \mathsf{R})$ then add the rule $bq' \to qa$.

Suppose M accepted input w, i.e., there are $\alpha, \beta \in \Gamma^*$ such that $q_0 w \vdash^* \alpha q_{\mathsf{acc}} \beta$. Then if the derivation guessed the correct accepting configuration using the first set of rules, and simulated correctly backwards this computation using the second set of rules, the derivation would have a string of the form $\rhd q_0 w \sqcup^i \lhd$ for some i. Now we will add rules to cleanup this string so that only the string w is generated. The rules to do that are as follows.

$$\sqcup \lhd \to \lhd \quad \lhd \to \epsilon \quad \rhd q_0 \to \epsilon$$

That completes the description of the grammar. The correctness is established by proving the following claim by induction on the number of steps (of the computation of M or the derivation, depending on the direction being proved),

$$q_0 w \vdash^* \alpha q_{\mathsf{acc}} \beta \text{ iff } S \overset{*}{\Rightarrow} \rhd \alpha q_{\mathsf{acc}} \beta \lhd \overset{*}{\Rightarrow} \rhd q_0 w \sqcup^i \lhd \overset{*}{\Rightarrow} w$$

□

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2 Restricted Grammars

2.1 Context Sensitive Grammars

Type 1 Grammars

The rules in a type 1 grammar are of the form

$$\alpha \to \beta$$

where $\alpha, \beta \in (\Sigma \cup V)^*$ *and* $|\alpha| \leq |\beta|$.

In every derivation, the length of the string never decreases.

Example 8. Consider the grammar G with $\Sigma = \{a, b, c\}$, $V = \{S, B, C, H\}$ and

$S \to aSBC \mid aBC$	$CB \to HB$	$HB \to HC$
$HC \to BC$	$aB \to ab$	$bB \to bb$
$bC \to bc$	$cC \to cc$	

$L(G) = \{a^n b^n c^n \mid n \geq 0\}$

Context Sensitivity

Normal Form for Type 1 grammars

For every Type 1 grammar G, there is a grammar (in normal form) G' such that $L(G) = L(G')$ and all the rules of G' are of the form

$$\alpha_1 A \alpha_2 \to \alpha_1 \beta \alpha_2$$

where $A \in V$ and $\beta \in (\Sigma \cup V)^*$

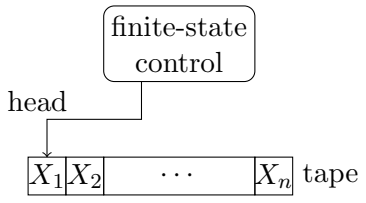
So, rules of G' replace a variable A by β in the context $\alpha_1 \square \alpha_2$. Thus, the class of language described by Type 1 grammars are called *context-sensitive languages*.

Expressive Power of Context Sensitive Languages

What languages can be described by Type 1 grammars?

- It turns out to be quite a lot!
- To say exactly, we need to define a new class of machines ...

Linear Bounded Automata



Definition 9. A *linear bounded automaton* is a restricted Turing machine where the tape head is not permitted to move beyond the portion of the tape containing the input.

- If the machine tries to move the head off either end of the input, the head stays where it is.

LBA and Type 1 Grammars

Theorem 10. *If G is a Type 1 grammar then there is a linear bounded automaton M such that $L(G) = L(M)$.*

If M is a linear bounded automaton then there is a Type 1 grammar G such that $L(M) = L(G)$.

Proof. Translations between TMs and Type 0 grammars, when carried out on Type 1 grammars and LBAs, prove this theorem. □

Decidability of LBAs

Theorem 11. *If M is a linear bounded automaton, then $L(M)$ is decidable.*

Proof.
- The number of configurations of M on an input of length n is at most snt^n, where s is the number of states of M and t is the size of the tape alphabet
  - Any configuration is state + head position + contents of the tape. The observation follows since the tape has at most n symbols.
- If M accepts w of length n then M does so within snt^n steps.
  - Any computation of length more than snt^n is "cycling" and so cannot accept w □

Decidability of LBAs

Proof (contd). Consider the following TM D that always halts and decides $L(M)$

```
On input w
    Run M on w for s|w|t^|w| steps
    If M accepts w then accept else reject
```

□

Model for Decidability?

Do LBAs recognize all decidable languages?

- LBAs recognize many but not all decidable languages.
- Decidable languages not recognized by LBAs can be found by diagonalization.

Diagonal LBA Language

Recall that every LBA can be coded as a binary string, and every binary string can be thought of as an LBA. We now consider only LBAs whose input alphabet is $\{0, 1\}$.

Theorem 12. *$L_{d,\text{LBA}} = \{M \mid M$ is a LBA and $M \notin L(M)\}$ is decidable but not context sensitive, i.e., recognized by an LBA.*

$L_{d,\text{LBA}}$ is decidable

The following program M decides $L_{d,\text{LBA}}$

```
On input x
    Check if x accepts x
    If x accepts x then reject else accept
```

Since languages recognized by LBAs are decidable, the step to check if x accepts x will halt.

$L_{d,\text{LBA}}$ is not context sensitive

- Suppose $L_{d,\text{LBA}}$ were recognized by a LBA, say M.
- Now, if $M \in L_{d,\text{LBA}} = L(M)$ then M is accepted by M, which means $M \notin L_{d,\text{LBA}}$!
- Conversely, if $M \notin L_{d,\text{LBA}} = L(M)$ then M is not accepted by M which means $M \in L_{d,\text{LBA}}$!

2.2 Regular Grammars

Type 3 Grammars

The rules in a type 3 grammar are of the form

$$A \to aB \qquad \text{or} \qquad A \to a$$

where $A, B \in V$ and $a \in \Sigma \cup \{\epsilon\}$.

Example 13. Consider the grammar over $\Sigma = \{0, 1\}$ with rules

$$S \to 1S \mid 0A \qquad A \to \epsilon \mid 1A \mid 0S$$

$L(G) = \{w \in \{0, 1\}^* \mid w$ has an odd number of 0s$\}$

Type 3 Grammars and Regularity

Proposition 14. *L is regular iff there is a Type 3 grammar G such that $L = L(G)$.*

Proof. Let $G = (V, \Sigma, R, S)$ be a type 3 grammar. Consider the NFA $M = (Q, \Sigma, \delta, q_0, F)$ where

- $Q = V \cup \{q_F\}$, where $q_F \notin V$
- $q_0 = S$
- $F = \{q_F\}$
- $\delta(A, a) = \{B \mid \text{if } A \to aB \in R\} \cup \{q_F \mid \text{if } A \to a \in R\}$ for $A \in V$. And $\delta(q_F, a) = \emptyset$ for all a.

$L(M) = L(G)$ as $\forall A \in V$, $\forall w \in \Sigma^*$, $A \stackrel{*}{\Rightarrow}_G w$ iff $q_F \in \hat{\Delta}(A, w)$. □

Type 3 Grammars and Regularity
NFA to Grammars

Proof (contd). Let $M = (Q, \Sigma, \delta, q_0, F)$ be a NFA recognizing L. Consider $G = (V, \Sigma, R, S)$ where

- $V = Q$
- $S = q_0$
- $q_1 \to aq_2 \in R$ iff $q_2 \in \delta(q_1, a)$ and $q \to \epsilon \in R$ iff $q \in F$.

We can show, for any $q, q' \in Q$ and $w \in \Sigma^*$, $q' \in \hat{\Delta}(q, w)$ iff $q \stackrel{*}{\Rightarrow}_G wq'$. Thus, $L(M) = L(G)$. □

2.3 Context Free Grammars

Type 2 Grammars

The rules in a type 2 grammar are of the form

$$A \to \beta$$

where $A \in V$ and $\beta \in (\Sigma \cup V)^*$.

Type 2 grammars describe *context-free languages*, which we will study next in this class.

Example 15. Consider G over $\Sigma = \{0, 1\}$ with rules

$$S \to \epsilon \mid 0S1$$

$L(G) = \{0^n 1^n \mid n \geq 0\}$

Grammar	Rules	Languages
Type 3	$A \to aB$ or $A \to a$	Regular
Type 2	$A \to \alpha$	Context Free
Type 1	$\alpha \to \beta$ with $\lvert\alpha\rvert \leq \lvert\beta\rvert$	Context Sensitive
Type 0	$\alpha \to \beta$	Recursively Enumerable

Figure 2: In the above table, $\alpha, \beta \in (\Sigma \cup V)^*$, $A, B \in V$ and $a \in \Sigma \cup \{\epsilon\}$.

3 Chomsky Hierarchy

Grammars and their Languages

Chomsky Hierarchy

Theorem 16. *Type 0, Type 1, Type 2, and Type 3 grammars define a* strict *hierarchy of formal languages.*

Proof. Clearly a Type 3 grammar is a special Type 2 grammar, a Type 2 grammar is a special Type 1 grammar, and a Type 1 grammar is special Type 0 grammar.

Moreover, there is a language that has a Type 2 grammar but no Type 3 grammar ($L = \{0^n 1^n \,|\, n \geq 0\}$), a language that has a Type 1 grammar but no Type 2 grammar ($L = \{a^n b^n c^n \,|\, n \geq 0\}$), and a language with a Type 0 grammar but no Type 1 grammar. □

Overview of Languages