# CS 373: Theory of Computation

Gul Agha Mahesh Viswanathan

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# 1 Undecidability

## 1.1 Recap

**Decision Problems and Languages**

- A *decision problem* requires checking if an input (string) has some property. Thus, a decision problem is a function from `strings` to `boolean`.
- A decision problem is represented as a *formal language* consisting of those strings (inputs) on which the answer is "yes".

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**Recursive Enumerability**

- A Turing Machine on an input $w$ either (halts and) accepts, or (halts and) rejects, or never halts.
- The language of a Turing Machine $M$, denoted as $L(M)$, is the set of all strings $w$ on which $M$ accepts.
- A language $L$ is *recursively enumerable/Turing recognizable* if there is a Turing Machine $M$ such that $L(M) = L$.

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**Decidability**

- A language $L$ is *decidable* if there is a Turing machine $M$ such that $L(M) = L$ and $M$ halts on every input.
- Thus, if $L$ is decidable then $L$ is recursively enumerable.

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**Undecidability**

**Definition 1.** A language $L$ is *undecidable* if $L$ is not decidable. Thus, there is no Turing machine $M$ that halts on every input and $L(M) = L$.

- This means that either $L$ is not recursively enumerable. That is there is no turing machine $M$ such that $L(M) = L$, or
- $L$ is recursively enumerable but not decidable. That is, any Turing machine $M$ such that $L(M) = L$, $M$ does not halt on some inputs.

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**Big Picture**

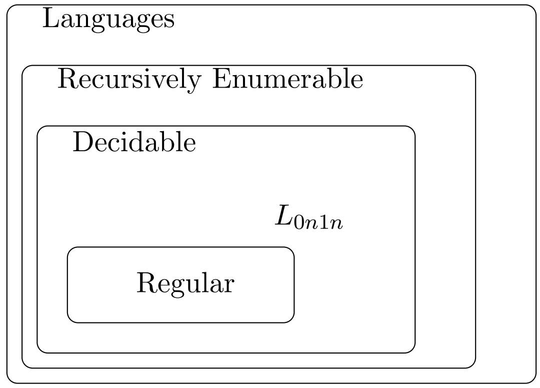


Figure 1: Relationship between classes of Languages

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**Machines as Strings**

- For the rest of this lecture, let us fix the input alphabet to be $\{0, 1\}$; a string over any alphabet can be encoded in binary.
- Any Turing Machine/program $M$ can itself be encoded as a binary string. Moreover every binary string can be thought of as encoding a TM/program. (If not the correct format, considered to be the encoding of a default TM.)
- We will consider decision problems (language) whose inputs are Turing Machine (encoded as a binary string)

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## 1.2 Diagonalization

**The Diagonal Language**

**Definition 2.** Define $L_d = \{M \mid M \notin L(M)\}$. Thus, $L_d$ is the collection of Turing machines (programs) $M$ such that $M$ does not halt and accept when given itself as input.

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**A non-Recursively Enumerable Language**

**Proposition 3.** *$L_d$ is not recursively enumerable.*

*Proof.* Recall that,

- Inputs are strings over $\{0, 1\}$
- Every Turing Machine can be described by a binary string and every binary string can be viewed as Turing Machine

- In what follows, we will denote the $i$th binary string (in lexicographic order) as the number $i$. Thus, we can say $j \in L(i)$, which means that the Turing machine corresponding to $i$th binary string accepts the $j$th binary string. □

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**Completing the proof**
*Diagonalization: Cantor*

*Proof (contd).* We can organize all programs and inputs as a (infinite) matrix, where the $(i, j)$th entry is Y if and only if $j \in L(i)$.

Inputs ⟶

| TMs ↓ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ⋯ |
|---|---|---|---|---|---|---|---|---|
| 1 | [N] | N | N | N | N | N | N | |
| 2 | N | [N] | N | N | N | N | N | |
| 3 | Y | N | [Y] | N | Y | Y | Y | |
| 4 | N | Y | N | [Y] | Y | N | N | |
| 5 | N | Y | N | Y | [Y] | N | N | |
| 6 | N | N | Y | N | Y | [N] | Y | |

Suppose $L_d$ is recognized by a Turing machine, which is the $j$th binary string. i.e., $L_d = L(j)$. But $j \in L_d$ iff $j \notin L(j)$! □

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**Acceptor for $L_d$?**

Consider the following program

```
On input i
    Run program i on i
    Output ``yes'' if i does not accept i
    Output ``no'' if i accepts i
```

Does the above program recognize $L_d$? No, because it may never output "yes" if $i$ does not halt on $i$.

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**Models for Decidable Languages**

**Question**
Is there a machine model such that

- all programs in the model halt on all inputs, and
- for each problem decidable by a TM, there is a program in the model that decides it?

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**Models for Decidable Languages**

**Answer**

There is no such model! Suppose there is a programming language in which all programs always halt. Programs in this language can be described by binary strings, and can be simulated by TMs. Consider the Turing Machine $M_d$

```
On input i
    Run program i on i
    Output ``yes'' if i does not accept i
    Output ``no'' if i accepts i
```

$M_d$ always halts and solves a problem not solved by any program in our language! Inability to halt is *essential* to capture all computation.

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## 1.3 The Universal Language

### Recursively Enumerable but not Decidable

- $L_d$ not recursively enumerable, and therefore not decidable. Are there languages that are recursively enumerable but not decidable?
- Yes, $A_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

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### The Universal Language

**Proposition 4.** *$A_{\text{TM}}$ is r.e. but not decidable.*

*Proof.* We have already seen that $A_{\text{TM}}$ is r.e. Suppose (for contradiction) $A_{\text{TM}}$ is decidable. Then there is a TM $M$ that always halts and $L(M) = A_{\text{TM}}$. Consider a TM $D$ as follows:

```
On input i
    Run M on input ⟨i,i⟩
    Output ``yes'' if i rejects i
    Output ``no'' if i accepts i
```

Observe that $L(D) = L_d$! But, $L_d$ is not r.e. which gives us the contradiction. □

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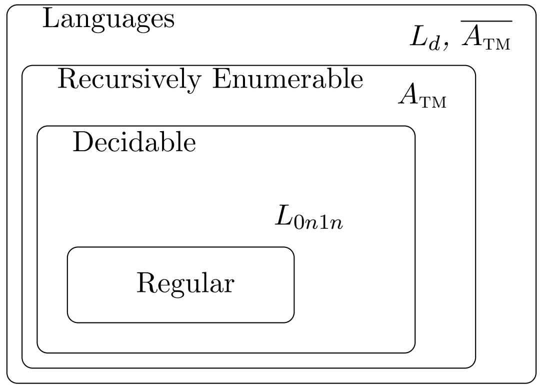
### A more complete Big Picture

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# 2 Reductions

## 2.1 Informal Overview

**Reductions**

A *reduction* is a way of converting one problem into another problem such that a solution to the second problem can be used to solve the first problem. We say the first problem *reduces* to the second problem.

- Informal Examples: Measuring the area of rectangle reduces to measuring the length of the sides; Solving a system of linear equations reduces to inverting a matrix
- The problem $L_d$ reduces to the problem $A_{\text{TM}}$ as follows: "To see if $w \in L_d$ check if $\langle w, w\rangle \in A_{\text{TM}}$."

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**Undecidability using Reductions**

**Proposition 5.** *Suppose $L_1$ reduces to $L_2$ and $L_1$ is undecidable. Then $L_2$ is undecidable.*

**Proof Sketch.**
Suppose for contradiction $L_2$ is decidable. Then there is a $M$ that always halts and decides $L_2$. Then the following algorithm decides $L_1$

- On input $w$, apply reduction to transform $w$ into an input $w'$ for problem 2
- Run $M$ on $w'$, and use its answer.

**Schematic View**

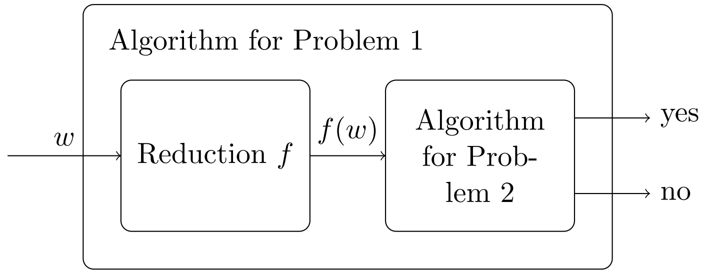


Figure 2: Reductions schematically

**The Halting Problem**

**Proposition 6.** *The language* $HALT = \{\langle M, w\rangle \mid M$ *halts on input* $w\}$ *is undecidable.*

*Proof.* We will reduce $A_{\text{TM}}$ to HALT. Based on a machine $M$, let us consider a new machine $f(M)$ as follows:

```
On input x
    Run M on x
    If M accepts then halt and accept
    If M rejects then go into an infinite loop
```

Observe that $f(M)$ halts on input $w$ if and only if $M$ accepts $w$ □

**The Halting Problem**
*Completing the proof*

*Proof (contd).* Suppose HALT is decidable. Then there is a Turing machine $H$ that always halts and $L(H) = \text{HALT}$. Consider the following program $T$

```
On input ⟨M, w⟩
    Construct program f(M)
    Run H on ⟨f(M), w⟩
    Accept if H accepts and reject if H rejects
```

$T$ decides $A_{\text{TM}}$. But, $A_{\text{TM}}$ is undecidable, which gives us the contradiction. □

## 2.2 Definition and Properties

### Mapping Reductions

**Definition 7.** A function $f : \Sigma^* \to \Sigma^*$ is *computable* if there is some Turing Machine $M$ that on every input $w$ halts with $f(w)$ on the tape.

**Definition 8.** A *mapping/many-one* reduction from $A$ to $B$ is a computable function $f : \Sigma^* \to \Sigma^*$ such that

$$w \in A \text{ if and only if } f(w) \in B$$

In this case, we say $A$ is *mapping/many-one reducible* to $B$, and we denote it by $A \leq_m B$.

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### Convention

In this course, we will drop the adjective "mapping" or "many-one", and simply talk about reductions and reducibility.

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### Reductions and Recursive Enumerability

**Proposition 9.** *If $A \leq_m B$ and $B$ is recursively enumerable then $A$ is recursively enumerable.*

*Proof.* Let $f$ be the reduction from $A$ to $B$ and let $M_B$ be the Turing Machine recognizing $B$. Then the Turing machine recognizing $A$ is

```
On input w
    Compute f(w)
    Run M_B on f(w)
    Accept if M_B does and reject if M_B rejects
```

□

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### Reductions and non-r.e.

**Corollary 10.** *If $A \leq_m B$ and $A$ is not recursively enumerable then $B$ is not recursively enumerable.*

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### Reductions and Decidability

**Proposition 11.** *If $A \leq_m B$ and $B$ is decidable then $A$ is decidable.*

*Proof.* Let $M_B$ be the Turing machine deciding $B$ and let $f$ be the reduction. Then the algorithm deciding $A$, on input $w$, computes $f(w)$ and runs $M_B$ on $f(w)$. □

**Corollary 12.** *If $A \leq_m B$ and $A$ is undecidable then $B$ is undecidable.*

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