

CS 373: Theory of Computation

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Context-Free Grammars

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- R is a finite set of rules or productions of the form $A \rightarrow \alpha$ where $A \in V$ and $\alpha \in (V \cup \Sigma)^*$
- $S \in V$ is the start symbol

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Or more briefly, $R = \{S \rightarrow \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}$

Language of a CFG

Derivations

Expand the start symbol using one of its rules. Further expand the resulting string by expanding one of the variables in the string, by the RHS of one of its rules. Repeat until you get a string of terminals.

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$G_{\text{pal}} = (\{S\}, \{0, 1\}, \{S \rightarrow \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}, S)$ we have

$$S \Rightarrow 0S0 \Rightarrow 00S00 \Rightarrow 001S100 \Rightarrow 0010100$$

Formal Definition

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Let $G = (V, \Sigma, R, S)$ be a CFG. We say $\alpha A \beta \Rightarrow_G \alpha \gamma \beta$, where $\alpha, \beta, \gamma \in (V \cup \Sigma)^*$ and $A \in V$ if $A \rightarrow \gamma$ is a rule of G .

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We say $\alpha \xRightarrow{*}_G \beta$ if either $\alpha = \beta$ or there are $\alpha_0, \alpha_1, \dots, \alpha_n$ such that

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Notation

When G is clear from the context, we will write \Rightarrow and $\xRightarrow{*}$ instead of \Rightarrow_G and $\xRightarrow{*}_G$.

Context-Free Language

Definition

The **language of CFG** $G = (V, \Sigma, R, S)$, denoted $L(G)$ is the collection of strings over the terminals derivable from S using the rules in R . In other words,

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Definition

A language L is said to be **context-free** if there is a CFG G such that $L = L(G)$.

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Recall, $L_{\text{pal}} = \{w \in \{0, 1\}^* \mid w = w^R\}$ is the language of palindromes.

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Proposition

$$L(G_{\text{pal}}) = L_{\text{pal}}$$

Proving Correctness of CFG

$$L_{\text{pal}} \subseteq L(G_{\text{pal}})$$

Proof.

Let $w \in L_{\text{pal}}$. We prove that $S \xRightarrow{*} w$

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- **Base Case:** If the derivation has only one step then the derivation must be $S \Rightarrow \epsilon$, $S \Rightarrow 0$ or $S \Rightarrow 1$. Thus $w = \epsilon$ or 0 or 1 and is in L_{Pal} .

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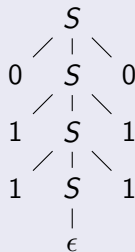
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Parse Trees

For CFG $G = (V, \Sigma, R, S)$, a **parse tree** (or **derivation tree**) of G is a tree satisfying the following conditions:

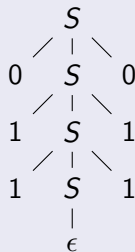


Example Parse Tree with yield
011110

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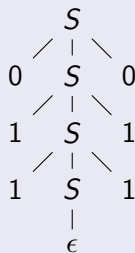


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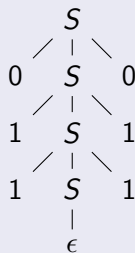


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- If an interior node labeled by A with children labeled by X_1, X_2, \dots, X_k (from the left), then $A \rightarrow X_1 X_2 \dots X_k$ must be a rule.

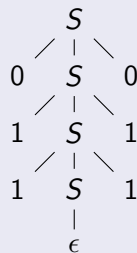


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Example Parse Tree with yield
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Yield of a parse tree is the concatenation of leaf labels (left–right)

Parse Trees and Derivations

Proposition

Let $G = (V, \Sigma, R, S)$ be a CFG. For any $A \in V$ and $\alpha \in (V \cup \Sigma)^$, $A \xRightarrow{*} \alpha$ iff there is a parse tree with root labeled A and whose yield is α .*

Parse Trees and Derivations

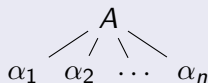
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Proof.

(\Rightarrow): Proof by induction on the number of steps in the derivation.

- **Base Case:** If $A \Rightarrow \alpha$ then $A \rightarrow \alpha$ is a rule in G . There is a tree of height 1, with root A and leaves the symbols in α .



\dashrightarrow

Parse Tree for Base Case

Parse Trees for Derivations

Proof (contd).

(\Rightarrow): Proof by induction on the number of steps in the derivation.

- **Induction Step:** Let $A \xRightarrow{*} \alpha$ in $k + 1$ steps.

Parse Trees for Derivations

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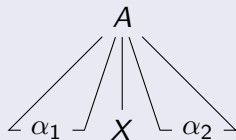
- **Induction Step:** Let $A \xRightarrow{*} \alpha$ in $k + 1$ steps.
- Then $A \xRightarrow{*} \alpha_1 X \alpha_2 \Rightarrow \alpha_1 \gamma \alpha_2 = \alpha$, where $X \rightarrow X_1 \cdots X_n = \gamma$ is a rule

Parse Trees for Derivations

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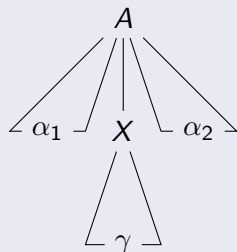
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- By ind. hyp., there is a tree with root A and yield $\alpha_1 X \alpha_2$.
- Add leaves X_1, \dots, X_n and make them children of X . New tree is a parse tree with desired yield. $\dots \rightarrow$



Parse Tree for Induction Step

Derivations for Parse Trees

Proof (contd).

(\Leftarrow): Assume that there is a parse tree with root A and yield α .
Need to show that $A \xRightarrow{*} \alpha$.



Derivations for Parse Trees

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Need to show that $A \xRightarrow{*} \alpha$. Proof by induction on the number of internal nodes in the tree.

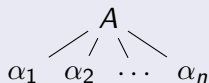


Derivations for Parse Trees

Proof (contd).

(\Leftarrow): Assume that there is a parse tree with root A and yield α . Need to show that $A \xRightarrow{*} \alpha$. Proof by induction on the number of internal nodes in the tree.

- **Base Case:** If tree has only one internal node, then it has the form as in picture



Parse Tree with one internal node

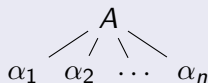


Derivations for Parse Trees

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- **Base Case:** If tree has only one internal node, then it has the form as in picture
- Then, $\alpha = X_1 \cdots X_n$ and $A \rightarrow \alpha$ is a rule. Thus, $A \xRightarrow{*} \alpha$.



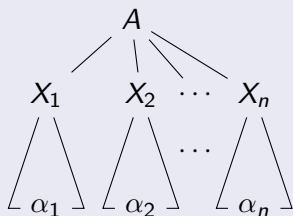
Parse Tree with one internal node



Derivations for Parse Trees

Proof (contd).

(\Leftarrow) **Induction Step:** Suppose α is the yield of a tree with $k + 1$ interior nodes. Let X_1, X_2, \dots, X_n be the children of the root ordered from the left. Not all X_i are leaves, and $A \rightarrow X_1 X_2 \cdots X_n$ must be a rule.

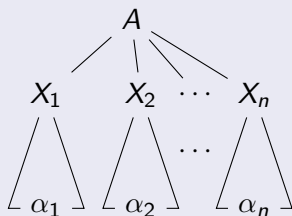
Tree with $k + 1$ internal nodes

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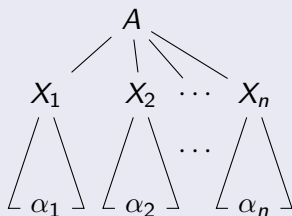
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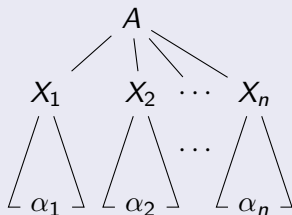
- Let α_i be the yield of the tree rooted at X_i ; so X_i is a leaf $\alpha_i = X_i$
- Now if $j < i$ then all the descendants of X_j are to the left of the descendants of X_i . So $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$.

Tree with $k + 1$ internal nodes

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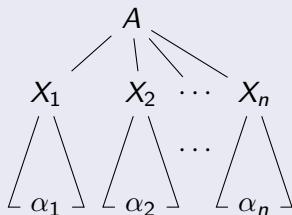


Derivations for Parse Trees

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(\Leftarrow) **Induction Step:** Suppose α is the yield of a tree with $k + 1$ interior nodes.

- Each subtree rooted at X_i has at most k internal nodes. So if X_i is a leaf $X_i \xRightarrow{*} \alpha_i$ and if X_i is not a leaf then $X_i \xRightarrow{*} \alpha_i$ (ind. hyp.).



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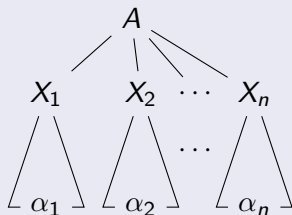
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- Thus

$$A \Rightarrow X_1 X_2 \cdots X_n \xRightarrow{*} \alpha_1 X_2 \cdots X_n \xRightarrow{*} \alpha_1 \alpha_2 \cdots X_n \xRightarrow{*} \alpha_1 \cdots \alpha_n = \alpha \quad \square$$



Recap ...

For a CFG G with variable A the following are equivalent

- 1 $A \xRightarrow{*} w$
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Context-free-ness

CFGs have the property that if $X \xRightarrow{*} \gamma$ then $\alpha X \beta \xRightarrow{*} \alpha \gamma \beta$

Example: English Sentences

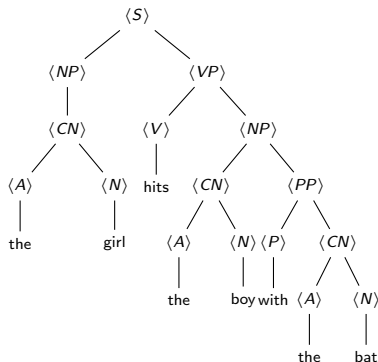
English sentences can be described as

$$\begin{aligned}\langle S \rangle &\rightarrow \langle NP \rangle \langle VP \rangle \\ \langle NP \rangle &\rightarrow \langle CN \rangle \mid \langle CN \rangle \langle PP \rangle \\ \langle VP \rangle &\rightarrow \langle CV \rangle \mid \langle CV \rangle \langle PP \rangle \\ \langle PP \rangle &\rightarrow \langle P \rangle \langle CN \rangle \\ \langle CN \rangle &\rightarrow \langle A \rangle \langle N \rangle \\ \langle CV \rangle &\rightarrow \langle V \rangle \mid \langle V \rangle \langle NP \rangle \\ \langle A \rangle &\rightarrow a \mid the \\ \langle N \rangle &\rightarrow boy \mid girl \mid bat \\ \langle V \rangle &\rightarrow hits \mid likes \mid sees \\ \langle P \rangle &\rightarrow with\end{aligned}$$

Multiple Parse Trees

Example 1

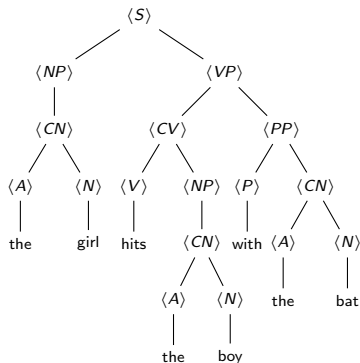
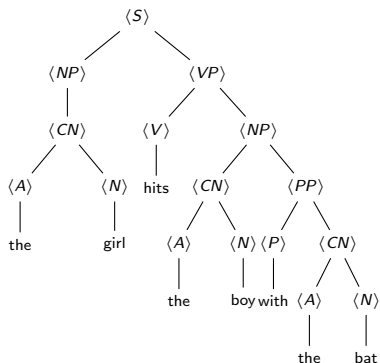
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Example: Arithmetic Expressions

Consider the language of all arithmetic expressions (E) built out of integers (N) and identifiers (I), using only $+$ and $*$

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Consider the language of all arithmetic expressions (E) built out of integers (N) and identifiers (I), using only $+$ and $*$

$G_{\text{exp}} = (\{E, I, N\}, \{a, b, 0, 1, (,), +, *, -\}, R, E)$ where R is

$$E \rightarrow I \mid N \mid -N \mid E + E \mid E * E \mid (E)$$

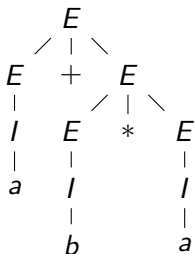
$$I \rightarrow a \mid b \mid Ia \mid Ib$$

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Multiple Parse Trees

Example 2

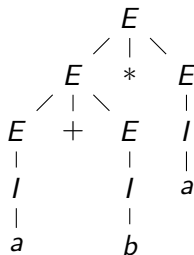
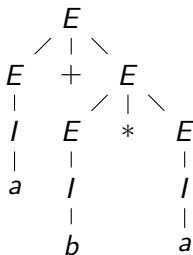
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Warning!

Existence of two derivations for a string does not mean the grammar is ambiguous!

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- Adding precedence to operators. For example, $*$ binds more tightly than $+$, or “else” binds with the innermost “if”.

An Example

Recall, G_{exp} has the following rules

$$E \rightarrow I \mid N \mid - N \mid E + E \mid E * E \mid (E)$$

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New CFG G'_{exp} has the rules

$$I \rightarrow a \mid b \mid Ia \mid Ib$$

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$$F \rightarrow I \mid N \mid - N \mid (E)$$

$$T \rightarrow F \mid T * F$$

$$E \rightarrow T \mid E + T$$

Ambiguity: Computational Problems

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Problem: Given CFG G , find CFG G' such that $L(G) = L(G')$ and G' is unambiguous.

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The problem is undecidable.

Problem: Is it the case that for every CFG G , there is a grammar G' such that $L(G) = L(G')$ and G' is unambiguous, *even if G' cannot be constructed algorithmically?*

Inherently Ambiguous Languages

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No! There are context-free languages L such that every grammar for L is ambiguous.

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Definition

A context-free language L is said to be **inherently ambiguous** if every grammar G for L is ambiguous.

Inherently Ambiguous Languages

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Consider

$$L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$$

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