CS 373: Theory of Computation

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Grammars

Definition

A grammar is $G = (V, \Sigma, R, S)$, where

- V is a finite set of variables/non-terminals
- Σ is a finite set of terminals
- $S \in V$ is the start symbol
- $R \subseteq (\Sigma \cup V)^* \times (\Sigma \cup V)^*$ is a finite set of rules/productions

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We say $\gamma_1 \alpha \gamma_2 \Rightarrow_G \gamma_1 \beta \gamma_2$ iff $(\alpha \to \beta) \in R$. And
$L(G) = \{w \in \Sigma^* \mid S \stackrel{*}{\Rightarrow}_G w\}$

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Consider the grammar G with $\Sigma = \{a\}$ with

$$
\begin{array}{lll}
S \rightarrow \$Ca\# \mid a \mid \epsilon & Ca \rightarrow aaC & \$D \rightarrow \$C \\
C\# \rightarrow D\# \mid E & aD \rightarrow Da & aE \rightarrow Ea \\
\$E \rightarrow \epsilon & &
\end{array}
$$

The following are derivations in this grammar

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S \Rightarrow \$Ca\# \Rightarrow \$aaC\# \Rightarrow \$aaE \Rightarrow \$aEa \Rightarrow \$Eaa \Rightarrow aa
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\begin{aligned}
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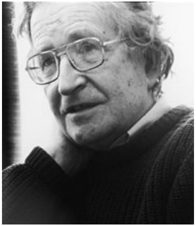
$L(G) = \{a^i \mid i \text{ is a power of } 2\}$

Grammars for each task

- What is the expressive power of these grammars?

Grammars for each task



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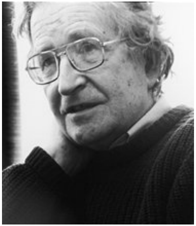

Noam Chomsky

- What is the expressive power of these grammars?
- Restricting the types of rules, allows one to describe different aspects of natural languages
- These grammars form a hierarchy

Type 0 Grammars

Definition

Type 0 grammars are those where the rules are of the form

$$\alpha \to \beta$$

where $\alpha, \beta \in (\Sigma \cup V)^*$

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Theorem

L is recursively enumerable iff there is a type 0 grammar G such that $L = L(G)$.

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L is recursively enumerable iff there is a type 0 grammar G such that $L = L(G)$.

Thus, type 0 grammars are as powerful as Turing machines.

Recognizing Type 0 languages

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If $G = (V, \Sigma, R, S)$ is a type 0 grammar then $L(G)$ is recursively enumerable.

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If $G = (V, \Sigma, R, S)$ is a type 0 grammar then $L(G)$ is recursively enumerable.

Proof.

We will show that $L(G)$ is recognized by a 2-tape non-deterministic Turing machine M, with tape 1 storing the input w, and tape 2 used to construct a derivation of w from S.

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Proof (contd).

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- If tape 2 contains only terminal symbols, then M will check to see if it matches tape 1. If so, the input is accepted, else it is rejected. $\square$

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Let M be a Turing machine recognizing L. The grammar G will simulate M "backwards" starting from an accepting configuration.

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- G has rules such that $\gamma_1 \Rightarrow \gamma_2$ iff $\gamma_2 \vdash_M \gamma_1$

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- Once (some) initial configuration $q_0 w$ is generated, rules in G will erase symbols to produce the terminal w.

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Details in the notes. □

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Example

Consider the grammar G with $\Sigma = \{a, b, c\}$, $V = \{S, B, C, H\}$ and

$S \to aSBC \mid aBC$	$CB \to HB$	$HB \to HC$
$HC \to BC$	$aB \to ab$	$bB \to bb$
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$L(G) = \{a^n b^n c^n \mid n \geq 0\}$

Context Sensitivity

Normal Form for Type 1 grammars

For every Type 1 grammar G, there is a grammar (in normal form) G' such that $L(G) = L(G')$ and all the rules of G' are of the form

$$\alpha_1 A \alpha_2 \rightarrow \alpha_1 \beta \alpha_2$$

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where $A \in V$ and $\beta \in (\Sigma \cup V)^*$
So, rules of G' replace a variable A by β in the context $\alpha_1 \Box \alpha_2$.
Thus, the class of language described by Type 1 grammars are called context-sensitive languages.

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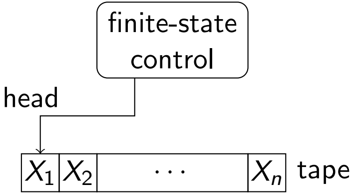
Expressive Power of Context Sensitive Languages

What languages can be described by Type 1 grammars?

- It turns out to be quite a lot!
- To say exactly, we need to define a new class of machines . . .

Linear Bounded Automata



Definition

A linear bounded automaton is a restricted Turing machine where the tape head is not permitted to move beyond the portion of the tape containing the input.

- If the machine tries to move the head off either end of the input, the head stays where it is.

LBA and Type 1 Grammars

Theorem

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Proof.

Translations between TMs and Type 0 grammars, when carried out on Type 1 grammars and LBAs, prove this theorem. $\square$

Decidability of LBAs

Theorem

If M is a linear bounded automaton, then $L(M)$ is decidable.

Proof.

- The number of configurations of M on an input of length n is at most snt^n, where s is the number of states of M and t is the size of the tape alphabet

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- The number of configurations of M on an input of length n is at most snt^n, where s is the number of states of M and t is the size of the tape alphabet
  - Any configuration is state + head position + contents of the tape. The observation follows since the tape has at most n symbols.
- If M accepts w of length n then M does so within snt^n steps.
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Decidability of LBAs

Proof (contd).

Consider the following TM D that always halts and decides $L(M)$

```
On input w
    Run M on w for s|w|t^|w| steps
    If M accepts w then accept else reject
```

∎

Model for Decidability?

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- Decidable languages not recognized by LBAs can be found by diagonalization.

Diagonal LBA Language

Recall that every LBA can be coded as a binary string, and every binary string can be thought of as an LBA. We now consider only LBAs whose input alphabet is $\{0, 1\}$.

Theorem

$L_{d,\mathrm{LBA}} = \{M \mid M$ *is a LBA and* $M \notin L(M)\}$ *is decidable but not context sensitive, i.e., recognized by an LBA.*

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The following program M decides $L_{d,\mathrm{LBA}}$

```
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Since languages recognized by LBAs are decidable, the step to check if x accepts x will halt.

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Type 3 Grammars

The rules in a type 3 grammar are of the form

$$A \to aB \qquad \text{or} \qquad A \to a$$

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Example

Consider the grammar over $\Sigma = \{0, 1\}$ with rules

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$L(G) = \{w \in \{0, 1\}^* \mid w \text{ has an odd number of 0s}\}$

Type 3 Grammars and Regularity

Proposition

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- $\delta(A, a) = \{B \mid \text{if } A \to aB \in R\} \cup \{q_F \mid \text{if } A \to a \in R\}$ for $A \in V$. And $\delta(q_F, a) = \emptyset$ for all a.

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$L(M) = L(G)$ as $\forall A \in V$, $\forall w \in \Sigma^*$, $A \stackrel{*}{\Rightarrow}_G w$ iff $q_F \in \hat{\Delta}(A, w)$.

Type 3 Grammars and Regularity

NFA to Grammars

Proof (contd).

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a NFA recognizing L. Consider $G = (V, \Sigma, R, S)$ where

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- $V = Q$

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- $q_1 \rightarrow aq_2 \in R$ iff $q_2 \in \delta(q_1, a)$ and $q \rightarrow \epsilon \in R$ iff $q \in F$.

□

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- $V = Q$
- $S = q_0$
- $q_1 \to aq_2 \in R$ iff $q_2 \in \delta(q_1, a)$ and $q \to \epsilon \in R$ iff $q \in F$.

We can show, for any $q, q' \in Q$ and $w \in \Sigma^*$, $q' \in \hat{\Delta}(q, w)$ iff $q \overset{*}{\Rightarrow}_G wq'$. Thus, $L(M) = L(G)$. $\square$

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Grammars and their Languages

Grammar	Rules	Languages
Type 3	$A \to aB$ or $A \to a$	Regular
Type 2	$A \to \alpha$	Context Free
Type 1	$\alpha \to \beta$ with $\lvert\alpha\rvert \leq \lvert\beta\rvert$	Context Sensitive
Type 0	$\alpha \to \beta$	Recursively Enumerable

In the above table, $\alpha, \beta \in (\Sigma \cup V)^*$, $A, B \in V$ and $a \in \Sigma \cup \{\epsilon\}$.

Chomsky Hierarchy

Theorem

Type 0, Type 1, Type 2, and Type 3 grammars define a strict *hierarchy of formal languages.*

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Type 0, Type 1, Type 2, and Type 3 grammars define a strict *hierarchy of formal languages.*

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Overview of Languages