CS 373: Theory of Computation

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Part I

Closure Properties of Turing Machines

Proposition

Decidable languages are closed under union, intersection, and complementation.

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Proof.

Given TMs M_1 , M_2 that decide languages L_1 , and L_2

• A TM that decides $L_1 \cup L_2$: on input x, run M_1 and M_2 on x, and accept iff either accepts.

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- A TM that decides L_1 : On input x, run M_1 on x, and accept if M_1 rejects, and reject if M_1 accepts.

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Decidable languages are closed under concatenation and Kleene Closure.

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We will show a decidable language L and a homomorphism h such that h(L) is undecidable

• Let $L = \{xy \mid x \in \{0,1\}^*, y \in \{a,b\}^*, x = \langle M,w \rangle$, and y encodes an integer n such that the TM M on input w will halt in n steps $\}$

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- h(L) = HALT which is undecidable.



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Complementation

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R.E. languages are not closed under complementation.

Proof.

 $A_{\rm TM}$ is r.e. but $\overline{A_{\rm TM}}$ is not.

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- A TM to recognize $h(L_1)$: On input x, start going through all strings w, and if h(w) = x, start executing M_1 on w, using dovetailing to interleave with other executions of M_1 . Accept if any of the executions accepts.

Part II

Grammars

Problem: Describe the set of arithmetic expressions with correctly matched parenthesis.

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Solution: Ignoring numbers and variables, and focussing only on parenthesis, correctly matched expressions can be defined as

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 - The concatenation of two correctly matched expressions, or
 - It must begin with (and end with) and moreover, once the first and last symbols are removed, the resulting string must correspond to a valid expression.

Grammar

Taking E to be the set of correct expressions, the inductive definition can be succinctly written as

$$E \rightarrow \epsilon$$

 $E \rightarrow EE$
 $E \rightarrow (E)$

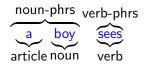
English Sentences

English sentences can be described as

$$\begin{array}{l} \langle S \rangle \rightarrow \langle NP \rangle \langle VP \rangle \\ \langle NP \rangle \rightarrow \langle CN \rangle \mid \langle CN \rangle \langle PP \rangle \\ \langle VP \rangle \rightarrow \langle CV \rangle \mid \langle CV \rangle \langle PP \rangle \\ \langle PP \rangle \rightarrow \langle P \rangle \langle CN \rangle \\ \langle CN \rangle \rightarrow \langle A \rangle \langle N \rangle \\ \langle CV \rangle \rightarrow \langle V \rangle \mid \langle V \rangle \langle NP \rangle \\ \langle A \rangle \rightarrow \text{a | the} \\ \langle N \rangle \rightarrow \text{boy | girl | flower} \\ \langle V \rangle \rightarrow \text{touches | likes | sees} \\ \langle P \rangle \rightarrow \text{with} \\ \end{array}$$

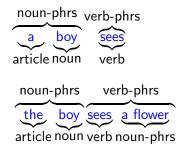
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Examples



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Applications

Such rules (or grammars) play a key role in

- Parsing programming languages and natural languages
- Markup Languages like HTML and XML.
- Modelling software

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- R is a finite set of rules or productions. Each production is of the form $\alpha \to \beta$ where $\alpha, \beta \in (V \cup \Sigma)^*$
- $S \in V$ is the start symbol; it is the variable that represents the language being defined. Other variables represent auxiliary languages that are used to define the language of the start symbol.

Example of a CFG

Example

Let $G_{par} = (V, \Sigma, R, S)$ be

- $V = \{E\}$
- $\Sigma = \{(,)\}$
- $R = \{E \rightarrow \epsilon, E \rightarrow EE, E \rightarrow (E)\}$
- \bullet S = E

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$$\begin{array}{l} S \rightarrow \epsilon \\ S \rightarrow 0 \\ S \rightarrow 1 \\ S \rightarrow 0S0 \\ S \rightarrow 1S1 \end{array}$$

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Or more briefly, $R = \{S \rightarrow \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}$

Arithmetic Expressions

Consider the language of all arithmetic expressions (E) built out of integers (N) and identifiers (I), using only + and *

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$$E \rightarrow I \mid N \mid E + E \mid E * E \mid (E)$$

 $I \rightarrow a \mid b \mid Ia \mid Ib$
 $N \rightarrow 0 \mid 1 \mid N0 \mid N1 \mid -N \mid +N$

More Examples

Example

Consider the grammar G with $\Sigma = \{a, b, c\}$, $V = \{S, B, C, H\}$ and

$$S \rightarrow aSBC \mid aBC$$

 $HC \rightarrow BC$

$$bC \rightarrow bc$$

$$\textit{CB} \rightarrow \textit{HB}$$

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Consider the grammar G with $\Sigma = \{a\}$ with

$$S \rightarrow Ca\# \mid a \mid \epsilon$$

$$C\# \rightarrow D\# \mid E$$

$$Ca \rightarrow aaC$$

 $aD \rightarrow Da$

$$D \rightarrow C$$
 $AE \rightarrow EA$

$$$E \rightarrow \epsilon$$

Derivation

Expand the start symbol using one of its rules. Then expand the resulting string by replacing one of its substrings that matches the LHS of a rule by the RHS. Repeat until you get a string of terminals.

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we have

$$E \Rightarrow E * E \Rightarrow E * N \Rightarrow E * -N \Rightarrow E * -N \Rightarrow E * -1$$

$$\Rightarrow (E) * -1 \Rightarrow (E + E) * -1 \Rightarrow (E + I) * -1$$

$$\Rightarrow (E + a) * -1 \Rightarrow (I + a) * -1 \Rightarrow (a + a) * -1$$

Formal Definition

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Let $G = (V, \Sigma, R, S)$ be a grammar. We say $\gamma_1 \alpha \gamma_2 \Rightarrow_G \gamma_1 \beta \gamma_1$, where $\gamma_1, \gamma_2 \alpha, \beta \in (V \cup \Sigma)^*$ if $\alpha \to \beta$ is a rule of G.

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We say $\alpha \stackrel{*}{\Rightarrow}_{G} \beta$ if either $\alpha = \beta$ or there are $\alpha_{0}, \alpha_{1}, \dots \alpha_{n}$ such that

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Notation

When G is clear from the context, we will write \Rightarrow and $\stackrel{*}{\Rightarrow}$ instead of \Rightarrow_G and $\stackrel{*}{\Rightarrow}_G$.

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$$L(G) = \{ w \in \Sigma^* \mid S \stackrel{*}{\Rightarrow} w \}$$

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Consider the grammar G with $\Sigma = \{a, b, c\}$, $V = \{S, B, C, H\}$ and

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$$L(G) = \{a^n b^n c^n \mid n \ge 0\}$$

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$$S \Rightarrow $Ca\# \Rightarrow $aaC\# \Rightarrow $aaD\# \Rightarrow $aDa\# \Rightarrow $Daa\# \Rightarrow $Caa\# \Rightarrow $aaCa\# \Rightarrow $aaaaC\# \Rightarrow $aaaaEa \Rightarrow $aaEaa \Rightarrow $aEaaa \Rightarrow $Eaaaa \Rightarrow aaaa$$

$$L(G) = \{a^i \mid i \text{ is a power of } 2\}$$