CS 373: Theory of Computation

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Regular Expressions and Regular Languages

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Theorem

L is a regular language if and only if there is a regular expression R such that $L(R) = L$

i.e., Regular expressions have the same "expressive power" as finite automata.

Proof.

- Given regular expression R, can construct NFA N such that $L(N) = L(R)$
- Given DFA M, will construct regular expression R such that $L(M) = L(R)$ □

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- Given DFA M, will construct regular expression R such that $L(M) = L(R)$. In two steps:
  - Construct a "Generalized NFA" (GNFA) G from the DFA M
  - And then convert G to a regex R

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    - For every pair of states (q_1, q_2), the transition from q_1 to q_2 is labeled by a regular expression $\rho(q_1, q_2)$.
  - "Generalized NFA" because a normal NFA has transitions labeled by ϵ, elements in Σ (a union of elements, if multiple edges between a pair of states) and $\emptyset$ (missing edges).

Generalized NFA

- Transition: GNFA non-deterministically reads a block of characters from the input, chooses an edge from the current state q_1 to another state q_2, and if the block of symbols matches the regex $\rho(q_1, q_2)$, then moves to q_2.

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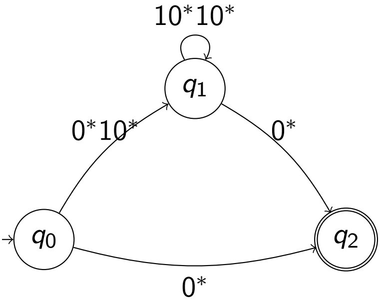
- Acceptance: G accepts w if there exists some sequence of valid transitions such that on starting from the start state, and after finishing the entire input, G is in the accept state.

Generalized NFA: Example



Example GNFA G

Accepting run of G on 11110100 is

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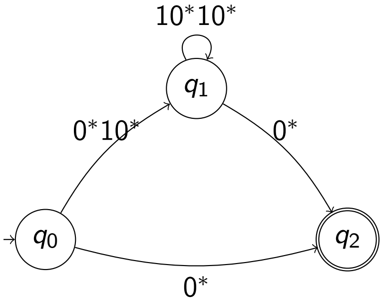

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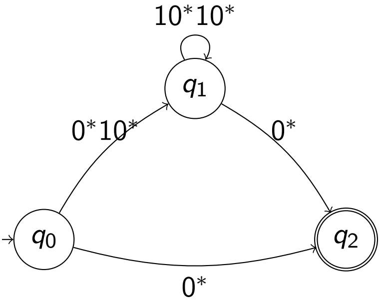

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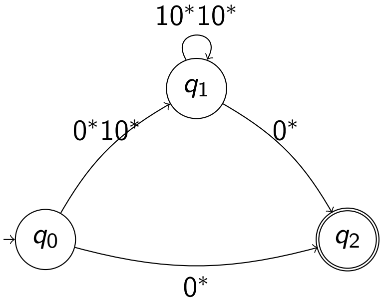

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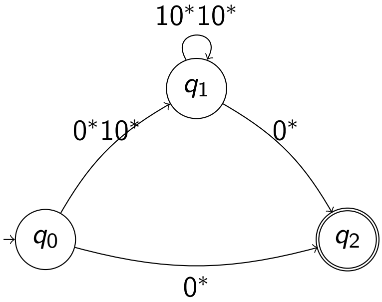

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A generalized nondeterministic finite automaton (GNFA) is $G = (Q, \Sigma, q_0, q_F, \rho)$, where

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- $q_F \in Q$, a single accepting state
- $\rho : (Q \setminus \{q_F\}) \times (Q \setminus \{q_0\}) \to \mathcal{R}_\Sigma$, where $\mathcal{R}_\Sigma$ is the set of all regular expressions over the alphabet Σ

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- $w = x_1 x_2 x_3 \cdots x_t$
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- for each $i \in [1, t]$, $x_i \in L(\rho(r_{i-1}, r_i))$,

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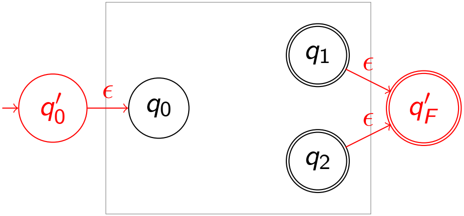
- $\rho(q_1, q_2) = \begin{cases} \epsilon, & \text{if } q_1 = q_0' \text{ and } q_2 = q_0 \\ \epsilon, & \text{if } q_1 \in F \text{ and } q_2 = q_F' \\ \bigcup_{\{a \mid \delta(q_1, a) = q_2\}} a & \text{otherwise} \end{cases}$



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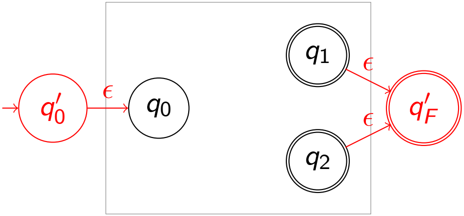

Prove: $L(G) = L(M)$.

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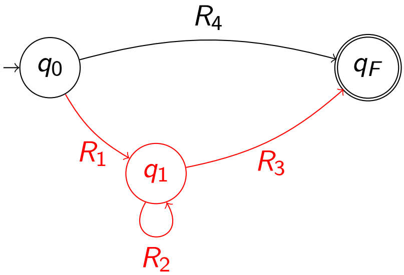

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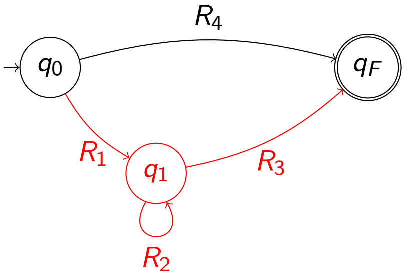

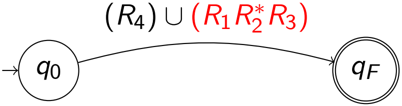

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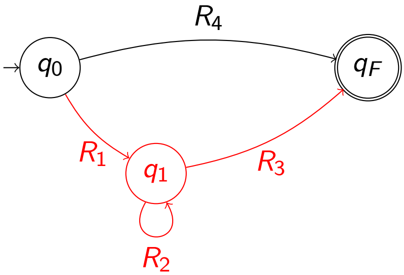

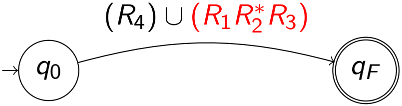

- Plan: Reduce any GNFA G with $k > 2$ states to an equivalent GFA with $k-1$ states.

GNFA to Regex: From k states to $k-1$ states

Definition (Deleting a GNFA State)

Given GNFA $G = (Q, \Sigma, q_0, q_F, \rho)$ with $|Q| > 2$, and any state $q^* \in Q \setminus \{q_0, q_F\}$, define GNFA $\mathrm{rip}(G, q^*) = (Q', \Sigma, q_0, q_F, \rho')$ as follows:

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  $$\rho'(q_1, q_2) = (R_1 R_2^* R_3) \cup R_4,$$

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Claim. For any $q^* \in Q \setminus \{q_0, q_F\}$, G and $\text{rip}(G, q^*)$ are equivalent.

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Proof.

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  - Case $a = b$. $(s_{j-1}, s_j) = (r_{b-1}, r_b)$ and $x_{[a,b]} = x_b \in L(R_4)$.
  - Case $a = b + 1 + u$. $x_a \in L(R_1)$, $x_{a+1}, \ldots, x_{b-1} \in L(R_2)$ and $x_b \in L(R_3)$. So $x_{[a,b]} \in L(R_1 R_2^u R_3)$.

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- Let $y_1, \ldots, y_d$ be the sequence of blocks of the form $x_{[a,b]}$.
- Then $w = y_1 \cdots y_d$ and $y_j \in L(\rho'(s_{j-1}, s_j))$.

GNFA to Regex: From k states to $k-1$ states

$w \in L(G) \implies w \in L(G')$

Proof.

- $w \in L(G) \implies w = x_1 x_2 x_3 \cdots x_t$, and a sequence of states $q_0 = r_0, r_1, \ldots, r_t = q_F$ s.t. $x_i \in L(\rho(r_{i-1}, r_i))$.
- Let $(q_0 = s_0, \ldots, s_d = q_F)$ be the subsequence of states obtained by deleting all occurrences of q^*.
- For any *run* of q^* — i.e., an interval $[a, b]$ s.t. $r_{a-1} \neq q^* = r_a = \ldots = r_{b-1} \neq r_b$ — let $x_{[a,b]} = x_a \cdots x_b$.
- If $s_{j-1} = r_{a-1}$ and $s_j = r_b$, then $x_{[a,b]} \in L(\rho'(s_{j-1}, s_j))$
- Let $y_1, \ldots, y_d$ be the sequence of blocks of the form $x_{[a,b]}$.
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i.e., $w \in L(G) \implies w \in L(G')$.

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(See notes for a formal argument.)

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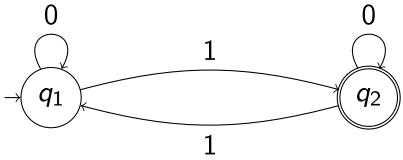
- For a 2-state GNFA G', $L(G') = L(R)$, where $R = \rho(q_0, q_F)$.

DFA to Regex: Example



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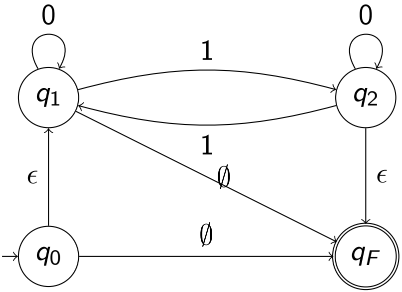

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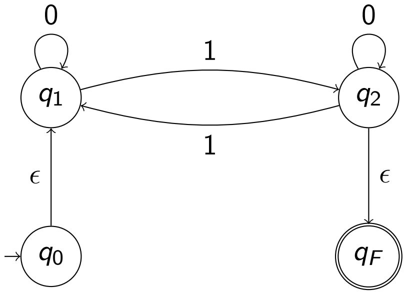

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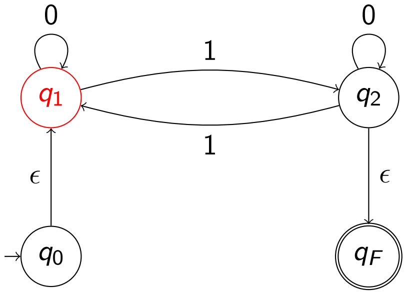

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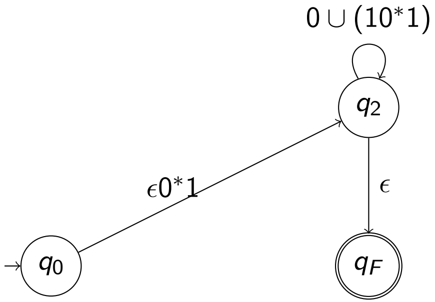

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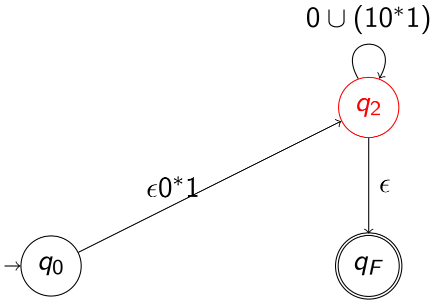

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