CS 373: Theory of Computation

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- A simple but powerful collection of operations:
  - Union, Concatenation and Kleene Closure

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L^*, the Kleene Closure of L: set of strings formed by taking any number of strings (possibly none) from L, possibly with repetitions and concatenating all of them.

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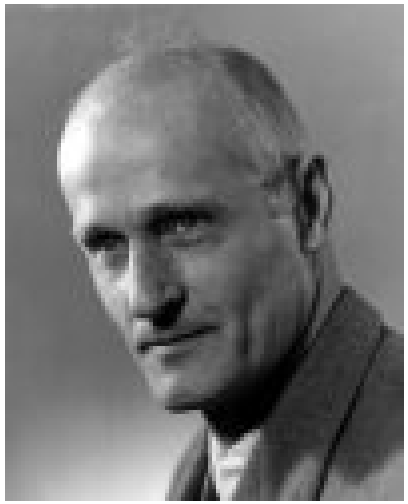
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Regular Expressions

A Simple Programming Language



Stephen Cole Kleene

A regular expression is a formula for representing a (complex) language in terms of "elementary" languages combined using the three operations union, concatenation and Kleene closure.

Regular Expressions

Formal Inductive Definition

Syntax and Semantics

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Also will sometimes omit $\circ$: e.g. will write RS instead of $R \circ S$

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Regular Expressions and Regular Languages

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- Given DFA M, will construct regular expression R such that $L(M) = L(R)$ □

Regular Expressions to Finite Automata

. . . to Non-determinstic Finite Automata

Lemma

For any regex R, there is an NFA N_R s.t. $L(N_R) = L(R)$.

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We will build the NFA N_R for R, inductively, based on the number of operators in R, $\#(R)$.

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- Induction Step: Consider R with $\#(R) = n + 1$. Based on the form of R, the NFA N_R will be built using the induction hypothesis.

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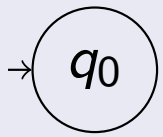
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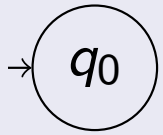


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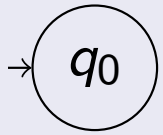
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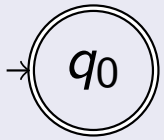
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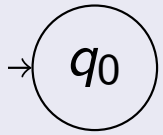


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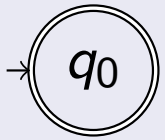
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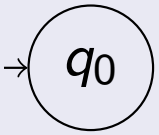
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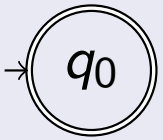
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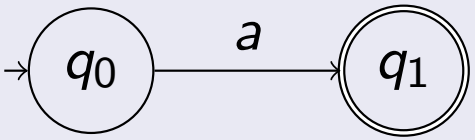
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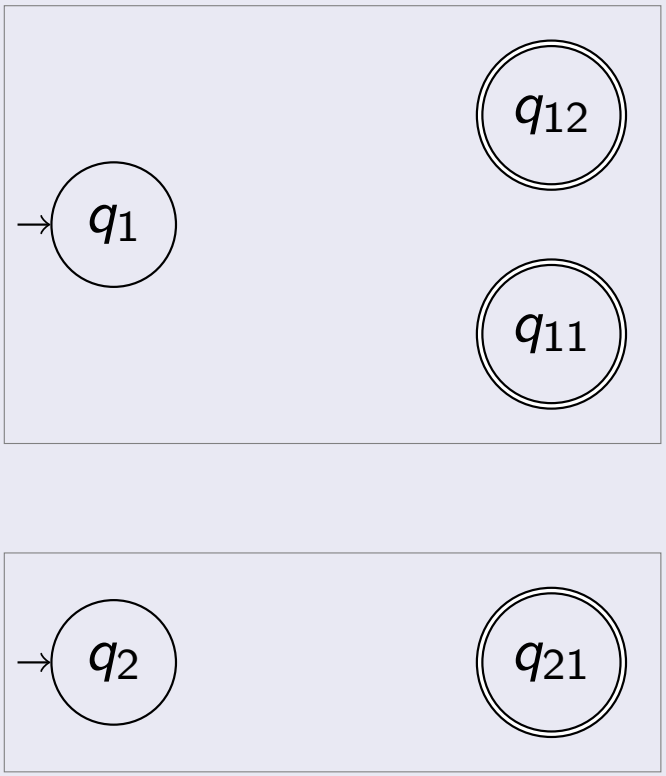
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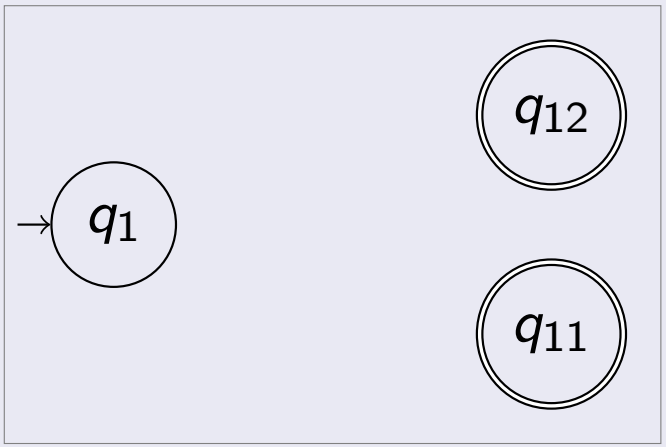
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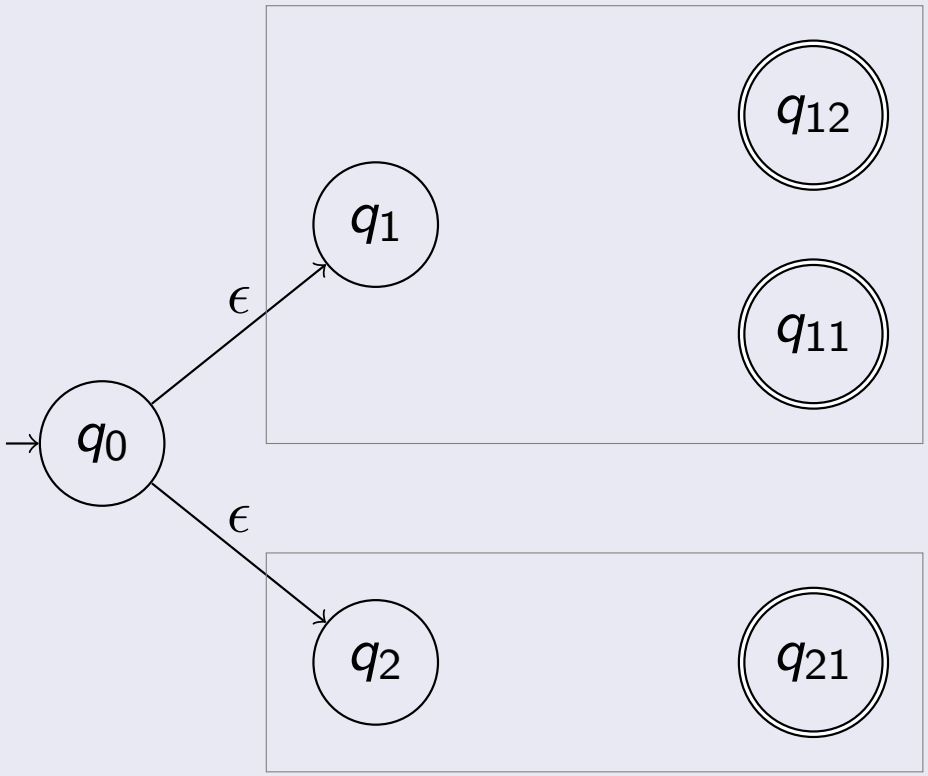




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Formal Definition

Case $R = R_1 \cup R_2$

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ (with $Q_1 \cap Q_2 = \emptyset$) such that $L(N_1) = L(R_1)$ and $L(N_2) = L(R_2)$. The NFA $N = (Q, \Sigma, \delta, q_0, F)$ is given by

- $Q = Q_1 \cup Q_2 \cup \{q_0\}$, where $q_0 \notin Q_1 \cup Q_2$
- $F = F_1 \cup F_2$
- δ is defined as follows

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q_1 \\ \delta_2(q, a) & \text{if } q \in Q_2 \\ \{q_1, q_2\} & \text{if } q = q_0 \text{ and } a = \epsilon \\ \emptyset & \text{otherwise} \end{cases}$$

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Correctness Proof

Need to show that $w \in L(N)$ iff $w \in L(N_1) \cup L(N_2)$.

$\Rightarrow$ $w \in L(N)$ implies $q_0 \xrightarrow{w}_N q$ for some $q \in F$.

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- $\Rightarrow$ $w \in L(N)$ implies $q_0 \xrightarrow{w}_N q$ for some $q \in F$. Based on the transitions out of q_0, $q_0 \xrightarrow{\epsilon}_N q_1 \xrightarrow{w}_N q$ or $q_0 \xrightarrow{\epsilon}_N q_2 \xrightarrow{w}_N q$. Consider $q_0 \xrightarrow{\epsilon}_N q_1 \xrightarrow{w}_N q$. (Other case is similar) This means $q_1 \xrightarrow{w}_{N_1} q$ (as N has the same transition as N_1 on the states in Q_1) and $q \in F_1$. This means $w \in L(N_1)$.

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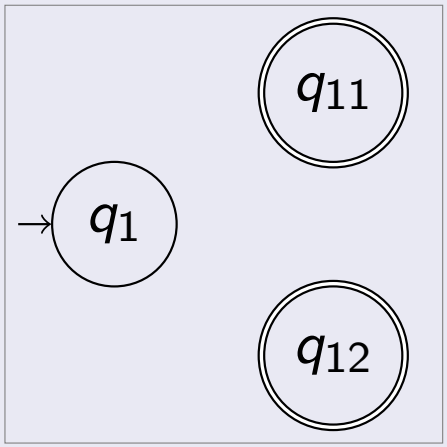
Induction Step: Concatenation

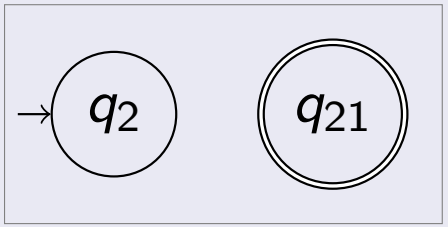
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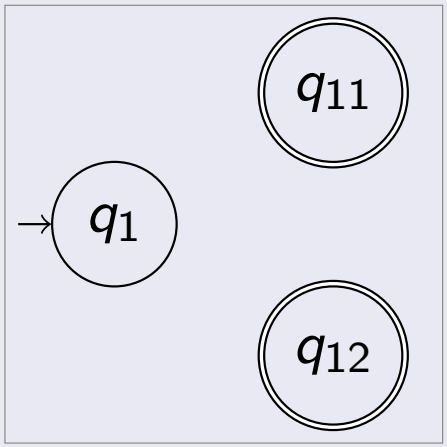


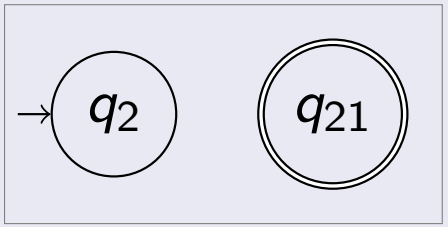


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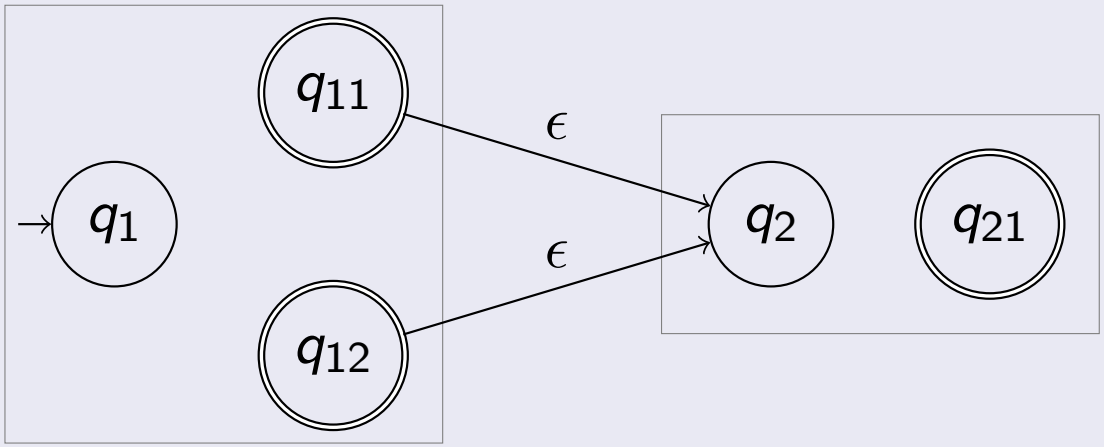




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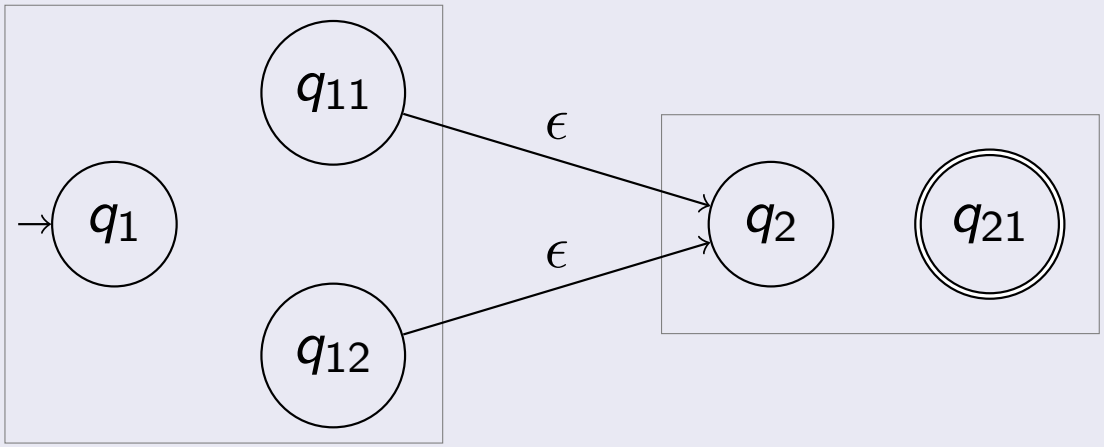
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Formal definition and proof of correctness left as exercise.

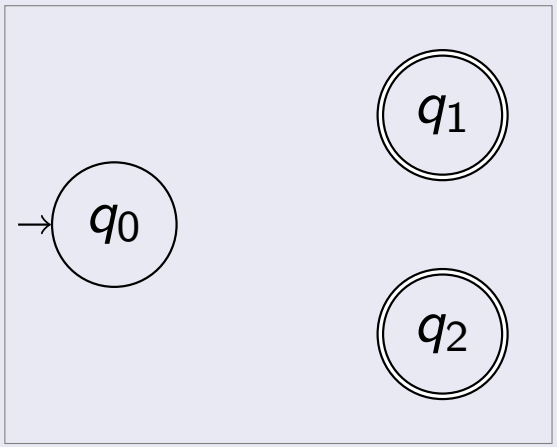
Induction Step: Kleene Closure

Case $R = R_1^*$

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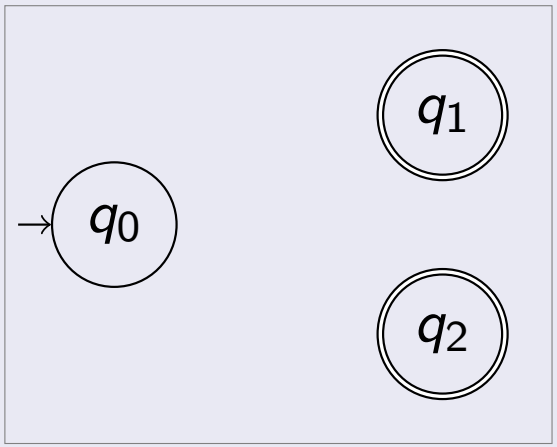
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Induction Step: Kleene Closure

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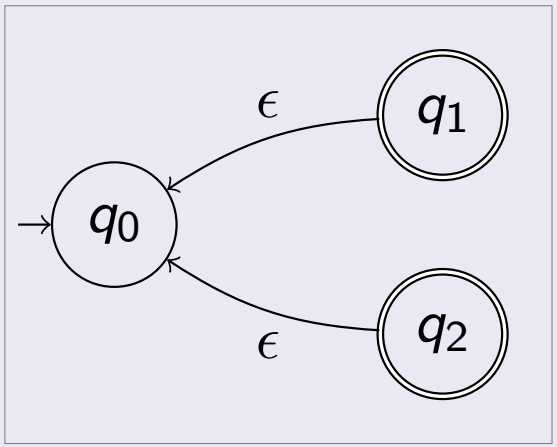
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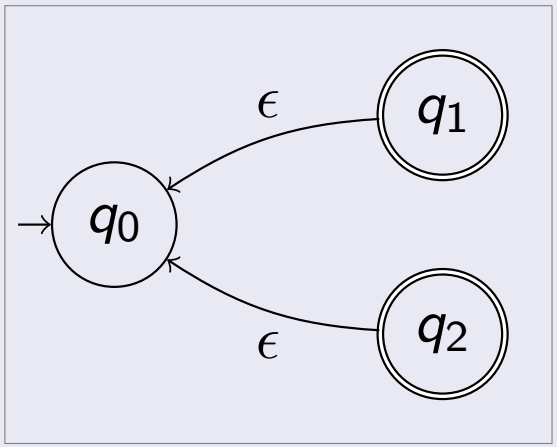


Induction Step: Kleene Closure

First Attempt

Case $R = R_1^*$

- By induction hypothesis, there is N_1 s.t. $L(N_1) = L(R_1)$
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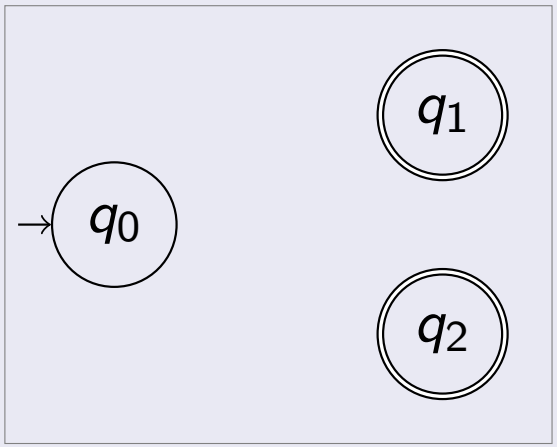


Problem: May not accept ϵ! One can show that $L(N) = (L(N_1))^+$.

Induction Step: Kleene Closure

Case $R = R_1^*$

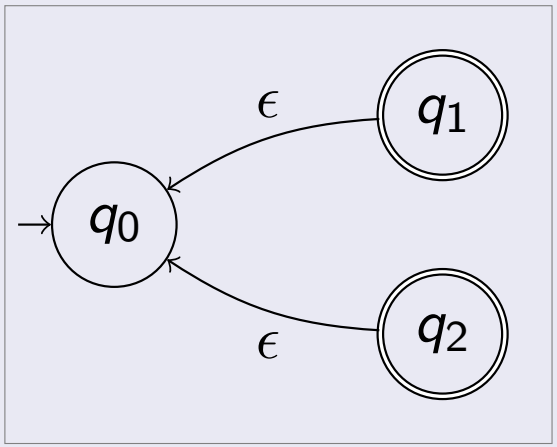
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Induction Step: Kleene Closure

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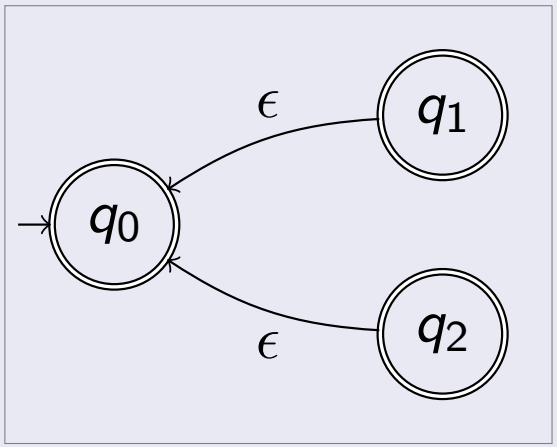
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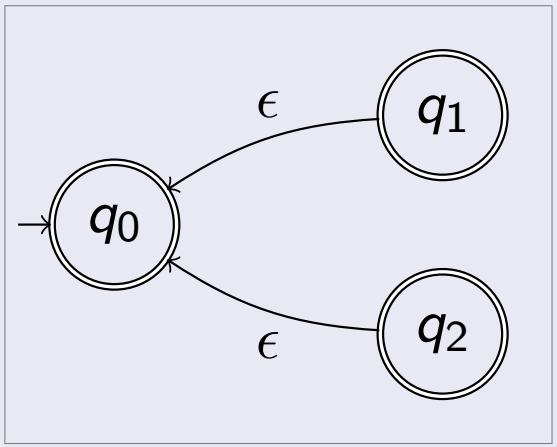


Induction Step: Kleene Closure

Second Attempt

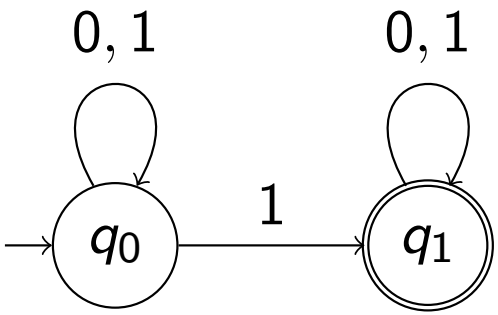
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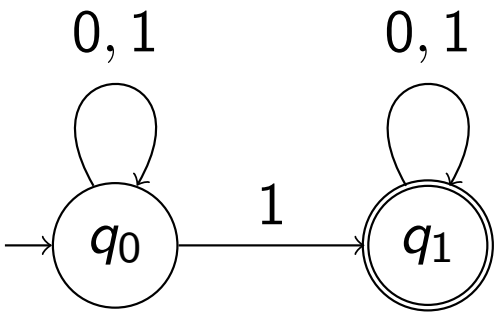
Problem: May accept strings that are not in $(L(N_1))^*$!

Example demonstrating the problem



Example NFA N

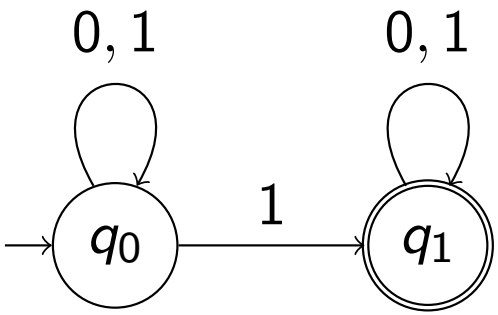
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Example NFA N

$L(N) = (0 \cup 1)^* 1 (0 \cup 1)^*$.

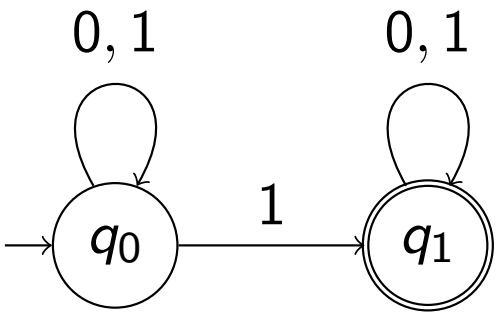
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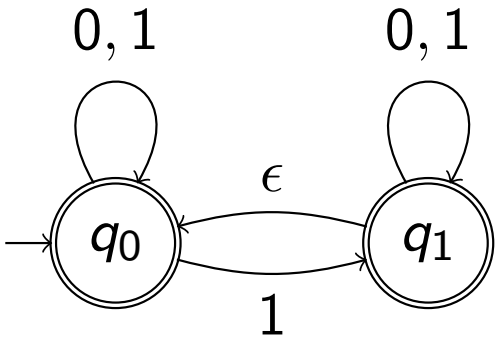
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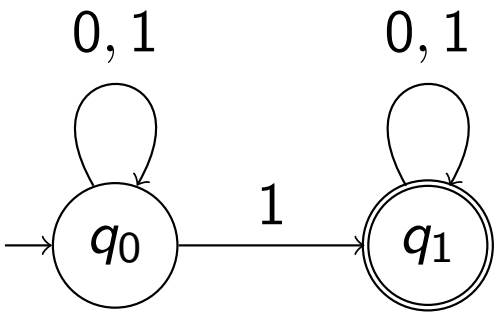
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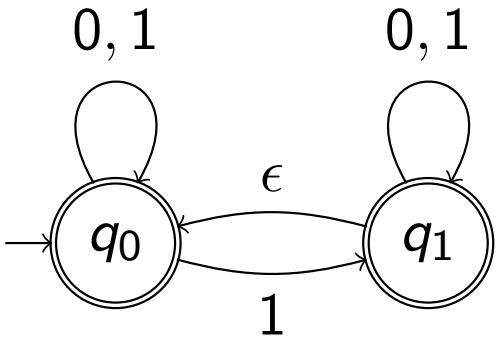
Incorrect Kleene Closure of N

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Example NFA N



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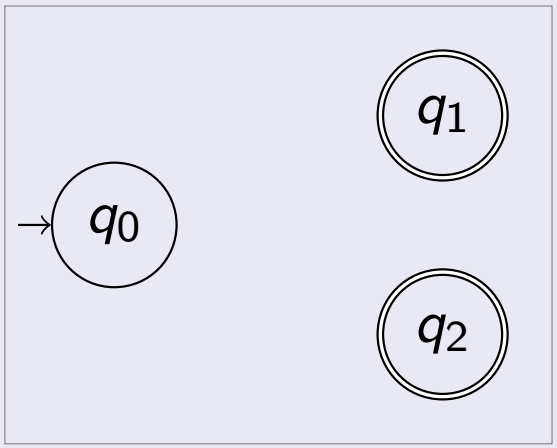
$L(N) = (0 \cup 1)^*1(0 \cup 1)^*$. Thus, $(L(N))^* = \epsilon \cup (0 \cup 1)^*1(0 \cup 1)^*$. The previous construction, gives an NFA that accepts $0 \notin (L(N))^*$!

Induction Step: Kleene Closure

Correct Construction

Case $R = R_1^*$

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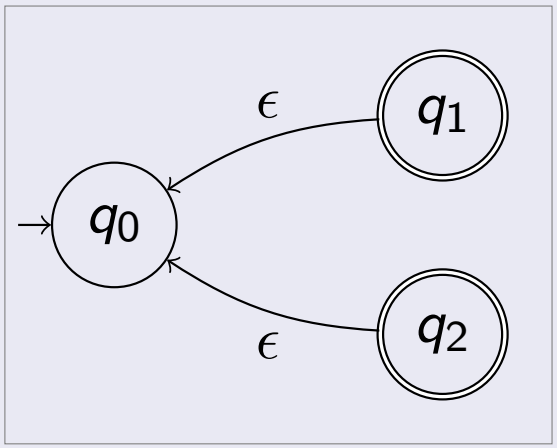


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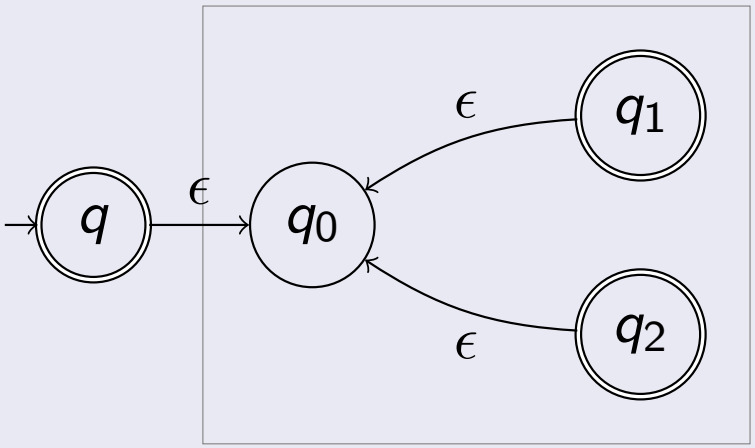


Induction Step: Kleene Closure

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Formal definition and proof of correctness left as exercise.

Regular Expressions to NFA

To Summarize

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- When R was an elementary regular expression, we gave an explicit construction of an NFA recognizing $L(R)$
- When $R = R_1 \operatorname{op} R_2$ (or $R = \operatorname{op}(R_1)$), we constructed an NFA N for R, using the NFAs for R_1 and R_2.

Regular Expressions to NFA

An Example

Build NFA for $(1 \cup 01)^*$

Regular Expressions to NFA

An Example

Build NFA for $(1 \cup 01)^*$

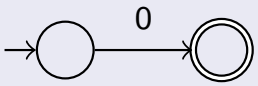
N_0

Regular Expressions to NFA

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N_0

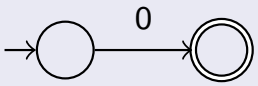


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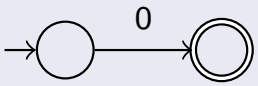


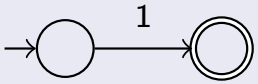
N_1

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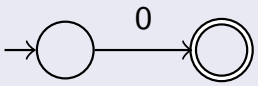
N_0 

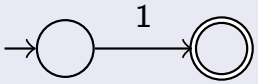
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Regular Expressions to NFA

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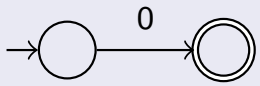
N_{01}

Regular Expressions to NFA

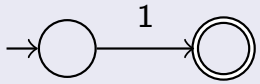
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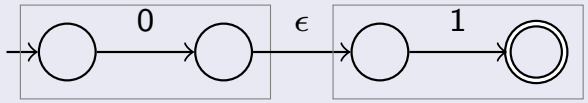
N_0



N_1



N_{01}

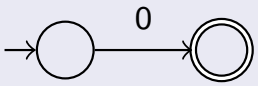


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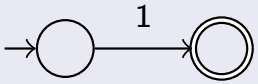
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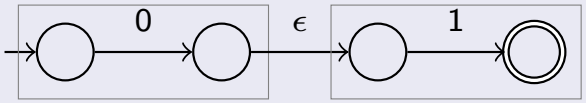
N_0



N_1



N_{01}



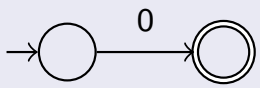
$N_{1 \cup 01}$

Regular Expressions to NFA

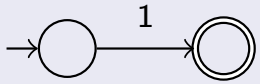
An Example

Build NFA for $(1 \cup 01)^*$

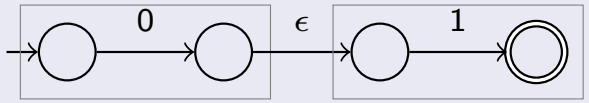
N_0



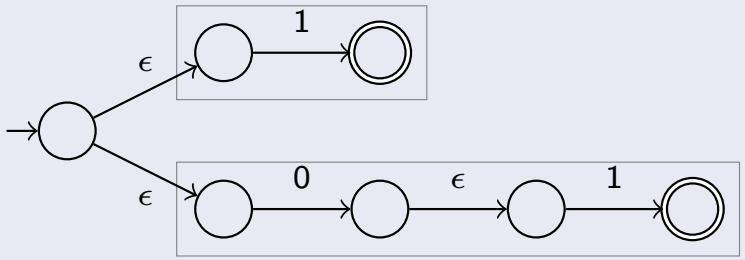
N_1



N_{01}



$N_{1 \cup 01}$



Example Continued

Build NFA for $(1 \cup 01)^*$

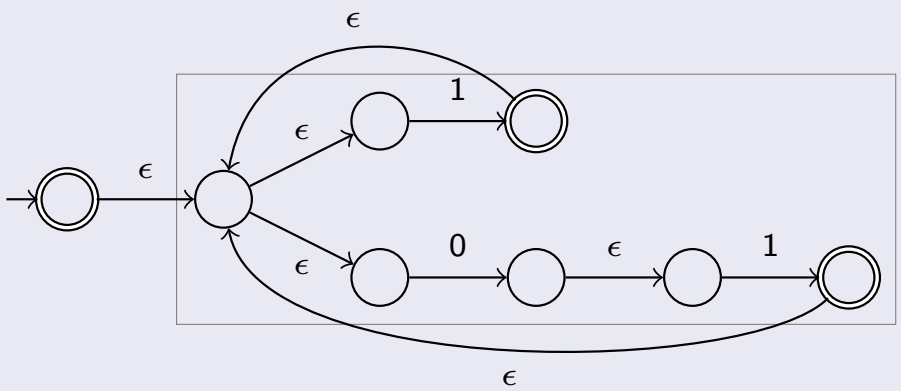
Example Continued

Build NFA for $(1 \cup 01)^*$

$N_{(1 \cup 01)^*}$

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$N_{(1 \cup 01)^*}$

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  - Today: Languages represented by regular expressions are regular (we showed how to build NFAs for them).
  - Coming up: Regular languages can be represented by regular expressions (by building regex for any given DFA).