
Solutions for Problem Set 5
CS 373: Theory of Computation

Assigned: October 5, 2010   Due on: October 12, 2010 at 10am

Homework Problems

Problem 1. [Category: Comprehension+Design] Consider the following DFA M. Let $L = L(M)$.

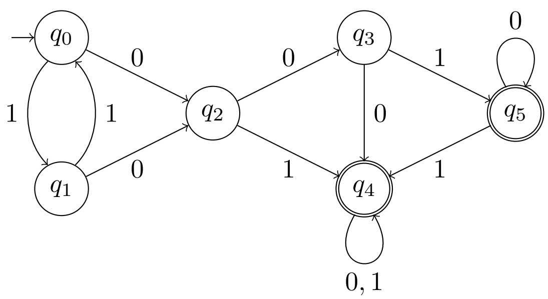


Figure 1: DFA M for Problem 1

1. What are the following sets: suffix(L, ϵ), suffix$(L, 0)$, suffix$(L, 00)$, and suffix$(L, 01)$? **[2 points]**
2. What are the following sets: suffix(M, q_0), suffix(M, q_2), suffix(M, q_3), and suffix(M, q_4)? **[2 points]**
3. Construct the minimum DFA that is equivalent to M, showing the steps of the construction clearly. **[3 points]**
4. For every pair of states in the minimal DFA constructed in the previous part, give a string that "distinguishes" the states. **[3 points]**

Solution:

1. suffix$(L, \epsilon) = L_1 = L(1^*0(00\cup01\cup1)(0\cup1)^*)$, suffix$(L, 0) = L_2 = L((00\cup01\cup1)(0\cup1)^*)$, suffix$(L, 00) = L_3 = L((0 \cup 1)(0 \cup 1)^*)$, suffix$(L, 01) = L_4 = (0\cup)^*$
2. suffix$(M, q_0) = L_1$, suffix$(M, q_2) = L_2$, suffix$(M, q_3) = L_3$, and suffix$(M, q_4) = L_4$, where L_1, L_2, L_3, and L_4 are the language defined in the previous part.
3. We will carry out the table filling algorithms outlined in class. In the tables below, if the entry corresponding to q_i and q_j is a, then it means that q_i and q_j are distinguishable, and when the transition labelled a is taken from q_i and q_j the go to states that are again distinguishable. We add these symbols to the table, so that information about the distinguishing string is in the table; we will see this in the next part.

   Observe that all the states are reachable, and so we don't remove any states before starting our algorithm. Initially the table looks like this. The entry ϵ indicating that the empty string distinguishes

	q_0	q_1	q_2	q_3	q_4
q_1					
q_2					
q_3					
q_4	ϵ	ϵ	ϵ	ϵ	
q_5	ϵ	ϵ	ϵ	ϵ	

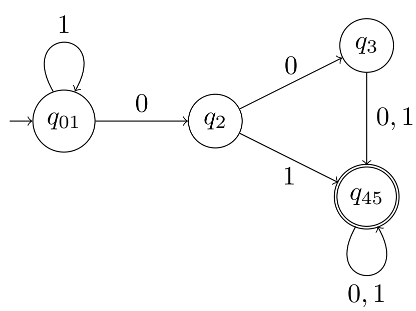
a final and non-final state.

In the next iteration the table looks like this.

	q_0	q_1	q_2	q_3	q_4
q_1					
q_2	1	1			
q_3	0	0	0		
q_4	ϵ	ϵ	ϵ	ϵ	
q_5	ϵ	ϵ	ϵ	ϵ	

Note, that entry 1 for the pair q_0 and q_2 indicates that on a 1 transition q_0 and q_2 go to states (q_1 and q_4, respectively) that were distinguishable in the previous iteration. This table does not change if we run it for one more iteration.

Thus, the minimum-state DFA is



4. The table filling algorithm described in the previous part already solves this part — if the entry for q_i and q_j is a then the string distinguishing q_i and q_j is aw, where w is (inductively) the string distinguishing $\delta(q_i, a)$ and $\delta(q_j, a)$. Let us see how this works.

   For any state $q \in \{q_0, q_1, q_2, q_3\}$ and $p \in \{q_4, q_5\}$ is (as the table says) ϵ. The string distinguishing q_2 and q_0 is $1w$, where w is the string distinguishing $\delta(q_0, 1) = q_1$ and $\delta(q_2, 1) = q_4$ (which is ϵ); thus, the string distinguishing q_0 and q_2 is 1. Similarly, for q_1 and q_2, the distinguishing string is 1. Finally, for $q \in \{q_0, q_1, q_2\}$ and q_3, the distinguishing string is 0.

■

Problem 2. [Category: Comprehension+Proof] For a string $w \in \Sigma^*$, let w^R denote the "reverse" of the string w. Consider $\text{Pal} = \{w \in \Sigma^* \mid w = w^R\}$; thus, Pal is the collection of *palindromes* over Σ. Prove that Pal is not regular using the Myhill-Nerode Theorem, by demonstrating that $\mathcal{C}_{\text{suf}}(\text{Pal})$ is infinite, i.e., there is an infinite set $W \subseteq \Sigma^*$ such that for any $x, y \in W$, $\text{suffix}(\text{Pal}, x) \neq \text{suffix}(\text{Pal}, y)$. **[10 points]**

Solution: Consider $W = \{0^i 1 \mid i \geq 0\}$; observe that W is an infinite set. Consider $0^i 1, 0^j 1 \in W$ such that $i \neq j$. Observe that $10^j \in \text{suffix}(\text{Pal}, 0^j 1)$ but $10^j \notin \text{suffix}(\text{Pal}, 0^i 1)$. Thus, $\text{suffix}(\text{Pal}, 0^i 1) \neq \text{suffix}(\text{Pal}, 0^j 1)$ and so $\mathcal{C}_{\text{suf}}(\text{Pal})$ is infinite. ■

Problem 3. [Category: Proof] For a language $L \subseteq \Sigma^*$ and string $x \in \Sigma^*$, define the *prefix language* of L with respect to x as

$$\text{prefix}(L, x) = \{y \mid yx \in L\}$$

Note, the difference between this and the way we defined $\text{suffix}(L, x)$ in class. Once again, the *class of prefix languages* (denoted as $\mathcal{C}_{\text{pref}}(L)$) of L is defined as

$$\mathcal{C}_{\text{pref}}(L) = \{\text{prefix}(L, x) \mid x \in \Sigma^*\}$$

Prove that L is regular if and only if $\mathcal{C}_{\text{pref}}(L)$ is finite. *Hint:* Can you see a connection between the prefix languages of L and the suffix languages of L^R? **[10 points]**

Solution: For a string $x \in \Sigma^*$, let $x^R \in \Sigma^*$ denote the reverse of string x. For a language $L \subseteq \Sigma^*$, let $L^R = \{x^R \mid x \in L\}$, be the "reverse" of language L. The crux of the proof is the following observation: $\text{prefix}(L, x) = (\text{suffix}(L^R, x^R))^R$. This can be shown as follows. Observe that $y \in \text{prefix}(L, x)$ if and only if $yx \in L$ (by definition of $\text{prefix}(L, \cdot)$) if and only if $x^R y^R \in L^R$ (since $(xy)^R = x^R y^R$) if and only if $y^R \in \text{suffix}(L^R, x^R)$ (by definition of $\text{suffix}(L^R, \cdot)$) if and only if $y \in (\text{suffix}(L^R, x^R))^R$.

We now use the previous observation to complete the proof. We showed in Problem Set 2, if L is regular then L^R is also regular. Moreover since $(L^R)^R = L$, we also have that if L^R is regular then L is regular. Thus, L is regular if and only if L^R is regular if and only if $\mathcal{C}_{\text{suf}}(L^R)$ is finite (by Myhill-Nerode theorem) if and only if $\{\text{suffix}(L^R, x) \mid x \in \Sigma^*\}$ is finite (by definition of $\mathcal{C}_{\text{suf}}(\cdot)$) if and only if $\{\text{suffix}(L^R, x^R) \mid x^R \in \Sigma^*\}$ is finite (since $\{x \in \Sigma^*\} = \{x^R \in \Sigma^*\}$) if and only if $\{(\text{suffix}(L^R, x^R))^R \mid x^R \in \Sigma^*\}$ is finite (since reverse is a one-to-one onto operation on languages) if and only if $\{\text{prefix}(L, x) \mid x \in \Sigma^*\}$ is finite (by observation proved) if and only if $\mathcal{C}_{\text{pref}}(L)$ is finite. ■