Solutions for Problem Set 4
CS 373: Theory of Computation

Assigned: September 21, 2010 Due on: September 28, 2010 at 10am

Homework Problems

Problem 1. [Category: Proof] Solve problem 1.49(b) using the pumping lemma.

Solution: Let p be the pumping length for C. Consider $w = 1^p01^p \in C$. Suppose x, y, z are such that $w = xyz$, with $|xy| \leq p$ and $|y| > 0$. Since, $|xy| \leq p$, without loss of generality, $x = 1^r$, $y = 1^s$, and $z = 1^t01^p$, with $r + s + t = p$ and $s > 0$.

Now, $xy^0z = 1^r\epsilon 1^t01^p = 1^{r+t}01^p$. Since $s > 0$, and $r + s + t = p$, we have $r + t < p$. Hence, $xy^0z \notin C$ and therefore, C does not satisfy the pumping lemma and is not regular. ■

Problem 2. [Category: Comprehension+Proof] Solve problem 1.54

Solution: Recall that the problem defines $F = \{a^ib^jc^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$.

a. Consider $A = F \cap L(ab^*c^*) = \{ab^nc^n \mid n \geq 0\}$. Define $h : \{a, b, c\}^* \to \{0, 1\}^*$ where $h(a) = \epsilon$, $h(b) = 0$ and $h(c) = 1$. Then, $h(A) = \{0^n1^n \mid n \geq 0\} = K$, which is known to be not regular. Thus, F is not regular as K was obtained from F by applying a series of regularity preserving operations.

b. Take the pumping length $p = 3$. Consider any $w = a^ib^jc^k \in F$, such that $|w| \geq p$.

   If $i \neq 2$, then divide w as follows: Take $x = \epsilon$, y to be the first symbol in w, and z to be the rest of the string. Now, $xyz = w$, $|xy| < 3$ and $|y| > 0$. Observe that the string xy^tz, when $t \neq 1$, has the property that the number of as is not 1, and hence $xy^tz \in L$ for any t.

   If $i = 2$, then divide w as follows: Take $x = aa$, y to be the first symbol after that, and z to be the rest of the string. Again, $w = xyz$, $|xy| \leq 3$, and $|y| > 0$. Further, for any t, xy^tz has 2 leading as, and so belongs to F trivially.

c. The pumping lemma says that every regular language satisfies the conditions in the pumping lemma, but does not say that *only* regular languages satisfy the pumping lemma. In particular, F is a non-regular language that satisfies the pumping lemma.

■

Problem 3. [Category: Design] Solve problem 1.41. You need not prove the correctness of your construction.

Solution: There are two possible ways this result can be proved: we can show that the perfect shuffle of A and B is regular by constructing a machine (DFA/NFA) recognizing it, or we can use closure properties. Here are both these styles of proofs.

Proof by Construction: Since A and B are regular, we know there are DFAs $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$ and $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$ recognizing A and B, respectively. The DFA M recognizing the perfect shuffle of A and B will run M_A and M_B alternately as it reads the symbols of the input. In order to do this, M

will need to remember the state of M_A reached on reading the "odd symbols" of the input so far, the state of M_B reached on reading the "even symbols" of the input so far, and which machine (M_A or M_B) is to be run next. Thus, M will be like a "modified" cross-product construction. The formal details are as follows.

Consider $M = (Q, \Sigma, \delta, q_0, F)$ such that

- $Q = Q_A \times Q_B \times \{A, B\}$; so a state of M is of the form (q_1, q_2, A) or (q_1, q_2, B) where q_1 is a state of Q_A and q_2 is a state of Q_B,
- $q_0 = (q_A, q_B, A)$; so initially we are in the initial states of M_A and M_B and the first symbol must be simulated on M_A,
- $F = F_A \times F_B \times \{A\}$; at the end, we must be in a final state of M_A and M_B, and the last symbol read must have been simulated on M_B (or the next symbol must be simulated on A,
- And δ is defined to ensure that in each step we simulated the machine whose turn it is, and the next turn belongs to the other machine. Formally,

$$\delta((q_1, q_2, T), a) = \begin{cases} (\delta_A(q_1, a), q_2, B) & \text{if } T = A \\ (q_1, \delta_B(q_2, a), A) & \text{if } T = B \end{cases}$$

The correctness proof (which we did not ask you to do) would proceed as follows. For $x = x_1 x_2 \cdots x_n \in \Sigma^*$ and $y = y_1 y_2 \cdots y_n \in \Sigma^*$ with $x_i, y_i \in \Sigma$, define $\text{perfect\_shuffle}(x, y) = x_1 y_1 x_2 y_2 \cdots x_n y_n$; note, here we are assuming that x and y are of (equal) length (say) n. To prove correctness we will show that for all $x, y \in \Sigma^*$ of equal length,

$$\hat{\delta}_M(q_0, \text{perfect\_shuffle}(x, y)) = (\hat{\delta}_{M_A}(q_A, x), \hat{\delta}_{M_B}(q_b, y), A)$$

This statement can be established by induction on the (common) length of x and y. Assuming this is proved, observe that $\text{perfect\_shuffle}(x, y) \in L(M)$ iff $\hat{\delta}_M(q_0, \text{perfect\_shuffle}(x, y)) \in F$ (defn. of acceptance) iff $\hat{\delta}_M(q_0, \text{perfect\_shuffle}(x, y)) = (q_1, q_2, A)$ and $q_1 \in F_A$ and $q_2 \in F_B$ (by defn. of F) iff $\hat{\delta}_{M_A}(q_A, x) = q_1$ and $q_1 \in F_A$ and $\hat{\delta}_{M_B}(q_B, y) = q_2 \in F_B$ (by statement of correctness) iff $x \in L(M_A) = A$ and $y \in L(M_B) = B$ (by defn. of acceptance). Thus, the correctness of the construction is established.

Proof of Closure Properties: Consider $\Delta = \Sigma \times \Sigma$. Define homomorphism $\text{left} : \Delta^* \to \Sigma^*$ as $\text{left}((a, b)) = a$. Consider $L_1 = \text{left}^{-1}(A)$; observe that $L_1 = \{(a_1, b_1)(a_2, b_2) \cdots (a_n, b_n) \mid n \geq 0,\ (a_i, b_i) \in \Delta,\ a_1 a_2 \cdots a_n \in A\}$.

Next, define homomorphism $\text{right} : \Delta^* \to \Sigma^*$ as $\text{right}((a, b)) = b$. Consider $L_2 = \text{right}^{-1}(B)$; observe that $L_2 = \{(a_1, b_1)(a_2, b_2) \cdots (a_n, b_n) \mid n \geq 0,\ (a_i, b_i) \in \Delta,\ b_1 b_2 \cdots b_n \in B\}$.

Observe that $L_3 = L_1 \cap L_2 = \{(a_1, b_1)(a_2, b_2) \cdots (a_n, b_n) \mid n \geq 0,\ (a_i, b_i) \in \Delta,\ a_1 a_2 \cdots a_n \in A,\ b_1 b_2 \cdots b_n \in B\}$. There is one final step to complete the proof. Consider the homomorphism $\text{unpair} : \Delta^* \to \Sigma^*$ where $\text{unpair}((a, b)) = ab$. Now, $\text{unpair}(L_3) = \text{perfect\_shuffle}(A, B)$, and so regular languages are closed under the perfect shuffle operation. Observe how short and clean the proof using closure propeties is as compared to the proof using explicit consructions. ■