Solutions for Problem Set 1
CS 373: Theory of Computation

Assigned: August 31, 2010 Due on: September 7, 2010

Homework Problems

Problem 1. [Category: Comprehension+Proof] Consider the following DFA M_0 over the alphabet $\{0, 1\}$.

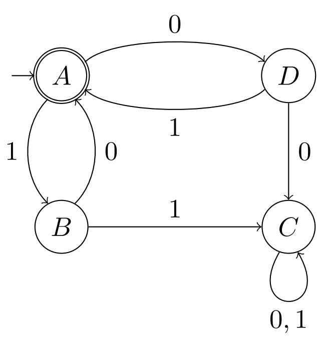

Figure 1: DFA M_0 for Problems 1 and 2

1. Describe formally what the following are for automaton M_0: set of states, initial state, final states, and transition function. **[2 points]**
2. What are $\hat{\delta}(A, \epsilon)$, $\hat{\delta}(A, 1011)$, $\hat{\delta}(B, 010)$, and $\hat{\delta}(C, 100)$? **[2 points]**
3. What is $L(M_0)$? Prove your answer. **[5 points]**
4. What is the language recognized if we change the initial state to B? What is the language recognized if we change the set of final states to be $\{B\}$ (with initial state A)? **[1 points]**

Solution:

1. States: $\{A, B, C, D\}$; Initial state: A; Final states: $\{A\}$; and transitions given by the following matrix

	0	1
A	D	B
B	A	C
C	C	C
D	C	D

2. $\hat{\delta}(A, \epsilon) = A$; $\hat{\delta}(A, 1011) = C$; $\hat{\delta}(B, 010) = A$; $\hat{\delta}(C, 100) = C$.
3. Let us call a string $w \in \{0, 1\}^*$ to be *proper* if in every prefix u of w has at most one more 0 than 1 and at most one more 1 than 0. Then

$$L(M_0) = \{w \in \{0, 1\}^* \mid w \text{ is proper and has equal number of 0s and 1s}\}$$

We will establish by induction on $|w|$ the following statements

(a) $\hat{\delta}(A, w) \in \{A\}$ iff $w \in L(M_0)$

(b) $\hat{\delta}(B, w) \in \{A\}$ iff $w = 0u$ where $u \in L(M_0)$

(c) $\hat{\delta}(C, w) \in \{A\}$ iff $w \in \emptyset$

(d) $\hat{\delta}(D, w) \in \{A\}$ iff $w = 1u$ where $u \in L(M_0)$

Base Case: Since $|w| = 0$, we know that $w = \epsilon$. Observe that $\epsilon \in L(M_0)$ and $\hat{\delta}(q, \epsilon) = q$ for any $q \in \{A, B, C, D\}$. Thus, $\hat{\delta}(q, \epsilon) \in \{A\}$ iff $q = A$, establishing all the four statements.

Induction Hypothesis: Assume that (a),(b),(c),(d) hold for strings w of length i.

Induction Step: Consider w of length $i+1$. Without loss of generality, we may assume that $w = av$, where $a \in \{0, 1\}$ and v is of length i. We have a few subcases to consider.

Subcase 1: Observe that $\hat{\delta}(A, 0v) = \hat{\delta}(\delta(A, 0), v)$ (by the proposition that we proved in class. Thus, we have

$$\begin{aligned}
\hat{\delta}(A, 0v) \in \{A\} &\text{iff } \hat{\delta}(D, v) \in \{A\} && (\delta(A, 0) = D) \\
&\text{iff } v = 1u \text{ where } u \in L(M_0) && (\text{ind. hyp.}) \\
&\text{iff } w = 0v \in L(M_0)
\end{aligned}$$

The other subcases are similar.

Subcase 2: Again, $\hat{\delta}(A, 1v) \in \{A\}$ iff $\hat{\delta}(B, v) \in \{A\}$ iff $v = 0u$ where $u \in L(M_0)$ iff $w = 0v \in L(M_0)$.

Subcase 3: $\hat{\delta}(B, 0v) \in \{A\}$ iff $\hat{\delta}(A, v) \in \{A\}$ iff $v \in L(M_0)$.

Subcase 4: $\hat{\delta}(B, 1v) \in \{A\}$ iff $\hat{\delta}(D, v) \in \{A\}$ iff $v \in \emptyset$.

Subcase 5: For any $a \in \{0, 1\}$, $\hat{\delta}(C, av) \in \{A\}$ iff $\hat{\delta}(C, v) \in \{A\}$ iff $v \in \emptyset$ iff $w = av \in \emptyset$.

Subcase 6: $\hat{\delta}(D, 0v) \in \{A\}$ iff $\hat{\delta}(C, v) \in \{A\}$ iff $v \in \emptyset$.

Subcase 7: $\hat{\delta}(D, 1v) \in \{A\}$ iff $\hat{\delta}(A, v) \in \{A\}$ iff $v \in L(M_0)$.

4. When the initial state is changed to B the language is

$$\{w \in \{0, 1\} \mid w = 1u \text{ where } u \in L(M_0)\}$$

Here $L(M_0)$ refers to the set defined in the previous part. When the set of final states is changed to $\{B\}$, the language is

$$\{w \in \{0, 1\}^* \mid w = u1 \text{ where } u \in L(M_0)\}$$

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Problem 2. [Category: Comprehension+Proof] Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$ define the following function $\rho : Q \times \Sigma^* \to 2^Q$ inductively. (Recall, 2^Q is the power set of Q.)

$$\rho(q, w) = \begin{cases} \{q\} & \text{if } w = \epsilon \\ \{q' \mid \delta(q', a) \in \rho(q, u)\} & \text{if } w = au \end{cases}$$

where $u \in \Sigma^*$ and $a \in \Sigma$. Answer the following questions about ρ and the DFA M_0 from problem 1.

1. What is $\rho(A, \epsilon)$, $\rho(A, 1011)$, $\rho(B, 010)$, and $\rho(C, 100)$? **[2 points]**

2. Give an english/mathematical description of what ρ is for a general DFA. **[1 points]**

3. For a DFA M, define $L'(M) = \{w \in \Sigma^* \mid \exists q \in F.\ q_0 \in \rho(q, w)\}$. For each of the following answer whether the belong to $L'(M_0)$: 0110, 101? **[1 points]**

4. What is $L'(M_0)$? **[1 points]**

5. For a general DFA M, what is the relationship between $L(M)$ and $L'(M)$? (Answer which of the following best describes the relationship: $L(M) = L'(M)$, $L(M) \subseteq L'(M)$ or $L'(M) \subseteq L(M)$.) Prove your answer. **[5 points]**

Solution:

1. $\rho(A, \epsilon) = \{A\}$; $\rho(A, 1011) = \emptyset$; $\rho(B, 010) = \emptyset$; $\rho(C, 100) = \{B, C, D\}$.

2. $\rho(q, w) = \{q' \in Q \mid \hat{\delta}(q', w) = q\}$

3. $0110 \in L'(M_0)$ and $101 \notin L(M_0')$

4. $L'(M_0) = L(M_0)$, where $L(M_0)$ is the set defined in the previous problem.

5. In general, $L'(M) = L(M)$. Let us assume that the definition of ρ given in part 2 is correct; we will prove this later by induction. Assuming that, we have

$$\begin{aligned} w \in L(M) &\text{iff } \hat{\delta}(q_0, w) \in F && \text{(defn. of } L(M)) \\ &\text{iff } \exists q \in F.\ \hat{\delta}(q_0, w) = q \\ &\text{iff } \exists q \in F.\ q_0 \in \rho(q, w) && \text{(part 2 to be proved)} \\ &\text{iff } w \in L'(M) && \text{(defn. of } L'(M)) \end{aligned}$$

Now we will prove for any $q \in Q$ and $w \in \Sigma^*$, $\rho(q, w) = \{q' \mid \hat{\delta}(q', w) = q\}$

Base Case: Consider w of length 0, i.e., $w = \epsilon$. $\rho(q, \epsilon) = \{q\} = \{q' \mid \hat{\delta}(q', \epsilon) = q\}$ as $\hat{\delta}(q', \epsilon) = q'$.

Induction Hypothesis: Assume that the observation holds for strings w of length i.

Induction Step: Consider $w = au$, where $|u| = i$ and $a \in \Sigma$.

$$\begin{aligned} \rho(q, au) = &\{q' \mid \delta(q', a) \in \rho(q, u)\} && \text{(defn. of } \rho) \\ &\{q' \mid \exists q''.\ \delta(q', a) = q'' \text{ and } \hat{\delta}(q'', u) = q\} && \text{(ind. hyp.)} \\ &\{q' \mid \hat{\delta}(\delta(q', a), u) = q\} \\ &\{q' \mid \hat{\delta}(q', au) = q\} && \text{(proposition about } \hat{\delta}) \end{aligned}$$

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Problem 3. [Category: Design] **[Modified version of problem 1.32 of text book]** Let

$$\Sigma_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ldots \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Σ_3 contains all size 3 columns of 0s and 1s. A string of symbols in Σ_3 gives 3 rows of 0s and 1s. Consider each row to be a binary number, where the first symbol is the least significant bit of each binary number. For example, the string

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

represents $0110 = 6$ (first row), $0011 = 3$ (second row) and $1001 = 9$ (third row). Let

$$B = \{w \in \Sigma_3 \mid \text{the bottom row of } w \text{ is the sum of the top two rows}\}$$

For example,

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in B \qquad \text{but} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \notin B$$

Design a DFA that recognizes B. You need not formally prove the correctness of your construction; however, your construction should be clear and understandable. **[10 points]**

Solution: We will assume that $\epsilon \in B$. The DFA M_B recognizing B will remember the "carry" from the previous bit position. Thus, there will be 2 states q_i (for $i\{0, 1\}$) denoting that the carry from the input seen so far is i. In addition, we will need to have an "error" state q_e to denote that an error in the sum has been discovered. So formally, $M_B = (Q, \Sigma_3, \delta, q_0, F)$ where

- $Q = \{q_0, q_1, q_e\}$,
- $F = \{q_0\}$, and
- δ is given as follows.

$$\delta(q_i, \begin{bmatrix} a \\ b \\ c \end{bmatrix}) = \begin{cases} q_{(a+b+i) \text{ div } 2} & \text{if } i \in \{0, 1\}, \text{ and } c = (a + b + i) \bmod 2 \\ q_e & \text{otherwise} \end{cases}$$

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