Problem Set 2

Fall 09

Due: Thursday Oct 1 at 11:00 AM in class (i.e., Room 103 Talbot Lab)

Please follow the homework format guidelines posted on the class web page:

`http://www.cs.uiuc.edu/class/fa09/cs373/`

1. Pumping Lemma. [**Points**: 10]
   Prove that these languages are not regular:

$$A = \{0^{n^3} \mid n \geq 0\}$$
$$B = \{0^n 1^m 0^k \mid 0 \leq n \leq m \leq k\}$$

Solution:

For A: we use contradiction, assume A is regular and p is its pumping length. Consider $w_p = 0^{p^3} \in A$. Assume $w_p = xyz = 0^{p^3}$ and $|xy| \leq p$, $|y| \neq 0$. Now we have $p^3 < |xy^2z| \leq p^3 + p < (p+1)^3$ which implies $xy^2z \notin A$, which contradicts the pumping lemma.

For B: we use contradiction, assume B is regular and p is its pumping length. Consider $w_p = 0^p 1^p 0^p \in B$. Assume $w_p = xyz = 0^p 1^p 0^p$ and $|xy| \leq p$, $|y| \neq 0$. Now xy is a substring of the p zeros to the left of w_p, so xy^2z will start with at least $p+1$ zeros followed by exactly p ones, which means $xy^2z \notin B$. This contradicts the pumping lemma.

2. Pumping Lemma. [**Points**: 10]
   Is the following language regular? Prove or disprove.

$$A = \{0^{\lfloor \lg^{100} n \rfloor} \mid n \geq 1\}$$

where $\lfloor x \rfloor$ is the largest integer not greater than x; and $\lg x$ is logarithm of x to the base 2.

Solution:

First we see that $\lim_{y \to \infty} 2^{\sqrt[100]{y+1}} - 2^{\sqrt[100]{y}} = +\infty$ (there is more than one way to see this, you may want to use your calculus knowledge to see it fast, our your general knowledge about how exponential function grows). This means that there is a fixed Y such that for all $y \geq Y$, we can find an integer x such that $2^{\sqrt[100]{y}} \leq x < 2^{\sqrt[100]{y+1}}$, which implies $y \leq \lg^{100} x < y+1$, which implies $\lfloor \lg^{100} x \rfloor = y$. In other words for all $y \geq Y$, we have $0^y \in A$. Let A_1 be the set of all strings in A of length less than Y. We have $A = A_1 \cup \{0^y \mid y \geq Y\}$. Both of these two sets is regular and the union of two regular sets remains regular, therefore A is a regular language.

3. Regular Expression. [**Points**: 10]
   Write regular expressions generating the following languages:

$$\begin{aligned} A =&\{w \in \{0,1\}^* \mid w \text{ has no substring } 011\} \\ B =&\{w \in \{0,1\}^* \mid |w| \text{ is a multiple of } 3\} \\ C =&\{w \in \{0,1\}^* \mid w \text{ has at least two 0 or at most one 1}\} \end{aligned}$$

Solution:

$$\begin{aligned} A =&L\big(1^*(0^* + 0^*01)\big) \\ B =&L\big(((0+1)(0+1)(0+1))^*\big) \\ C =&L\big(1^*01^*0(0+1)^* + 0^*10^* + 0 + \epsilon\big) \end{aligned}$$

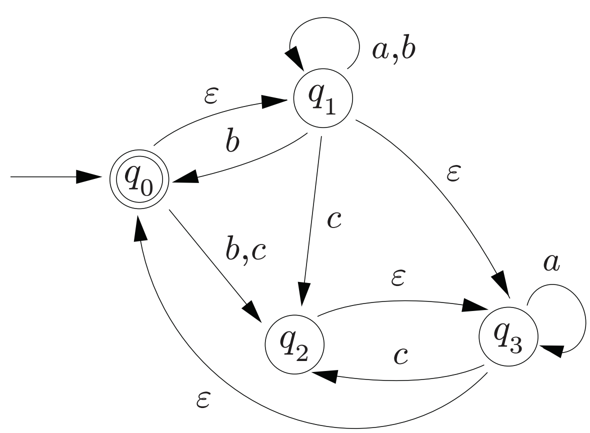
4. Regular Expression. [**Points**: 10]
   Prove or disprove: For languages A and B of the same alphabet Σ, if AB and B are regular, then A must be regular.

Solution:

This is wrong. Consider $A = \{0^n1^m \mid m \leq n\}, B = L(1^*)$ and observe that $AB = L(0^*1^*)$.

5. ϵ-closure [**Points**: 10]
   Compute the ϵ-closure for the following NFAs

   (a) $\Sigma = \{a, b, c\}$:

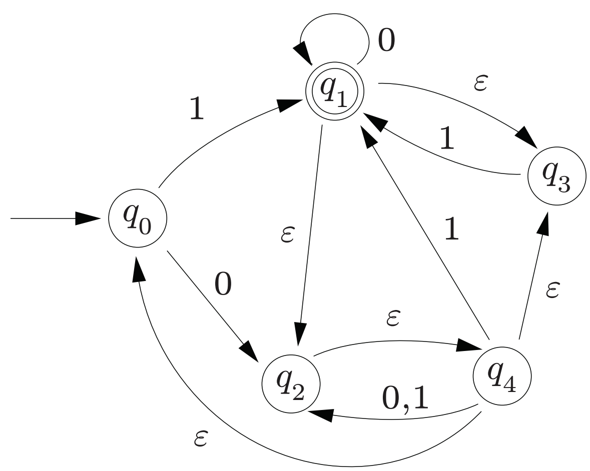


Solution:

We give the transition function for the new NFA with the same states. Note that the δ function is highly symmetric.

δ	a	b	c
q_0	$\{q_0, q_1, q_3\}$	$\{q_0, q_1, q_2, q_3\}$	$\{q_0, q_1, q_2, q_3\}$
q_1	$\{q_0, q_1, q_3\}$	$\{q_0, q_1, q_2, q_3\}$	$\{q_0, q_1, q_2, q_3\}$
q_2	$\{q_0, q_1, q_3\}$	$\{q_0, q_1, q_2, q_3\}$	$\{q_0, q_1, q_2, q_3\}$
q_3	$\{q_0, q_1, q_3\}$	$\{q_0, q_1, q_2, q_3\}$	$\{q_0, q_1, q_2, q_3\}$

(b) $\Sigma = \{0, 1\}$:



Solution:

We give the transition function for the new NFA with the same states.

δ	0	1
q_0	$\{q_0, q_2, q_3, q_4\}$	$\{q_0, q_1, q_2, q_3, q_4\}$
q_1	$\{q_0, q_1, q_2, q_3, q_4\}$	$\{q_0, q_1, q_2, q_3, q_4\}$
q_2	$\{q_0, q_2, q_3, q_4\}$	$\{q_0, q_1, q_2, q_3, q_4\}$
q_3	$\{\}$	$\{q_0, q_1, q_2, q_3, q_4\}$
q_4	$\{q_0, q_2, q_3, q_4\}$	$\{q_0, q_1, q_2, q_3, q_4\}$

6. DFA minimization [**Points**: 10]

(a) Determine the minimal-size DFA M such that $L(M) = L(M_1)$, $M_1 = (\Sigma, \delta, Q, q_0, F)$:
$\Sigma = \{0, 1\}$,
$Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7\}$,
$q_{start} = q_0$,
$F = \{q_3\}$,
and δ, the transition function

	0	1
q_0	q_1	q_0
q_1	q_0	q_2
q_2	q_3	q_1
q_3	q_3	q_0
q_4	q_3	q_5
q_5	q_6	q_4
q_6	q_5	q_6
q_7	q_6	q_3

Solution:

The new machine is basically the old one with q_4, q_5, q_6, q_7 removed.
$M_1' = (\Sigma, \delta, Q, q_0, F)$:
$\Sigma = \{0, 1\}$,
$Q = \{q_0, q_1, q_2, q_3\}$,
$q_{start} = q_0$,
$F = \{q_3\}$,
and δ, the transition function

	0	1
q_0	q_1	q_0
q_1	q_0	q_2
q_2	q_3	q_1
q_3	q_3	q_0

(b) Determine the minimal-size DFA M such that $L(M) = L(M_2)$, $M_2 = (\Sigma, \delta, Q, q_0, F)$:
$\Sigma = \{0, 1\}$,
$Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7\}$,
$q_{start} = q_0$,
$F = \{q_5, q_6\}$,
and δ, the transition function

	0	1
q_0	q_7	q_1
q_1	q_7	q_0
q_2	q_4	q_5
q_3	q_4	q_5
q_4	q_5	q_6
q_5	q_5	q_5
q_6	q_6	q_5
q_7	q_2	q_2

Solution:

$q_0/q_1, q_2/q_3, q_5/q_6$ are equivalent. Take these as new combined states. We have
$M_2' = (\Sigma, \delta, Q, q_0/q_1, F)$:
$\Sigma = \{0, 1\}$,
$Q = \{q_0/q_1, q_2/q_3, q_4, q_5/q_6, q_7\}$,
$q_{start} = q_0$,
$F = \{q_5, q_6\}$,
and δ, the transition function

	0	1
q_0/q_1	q_7	q_0/q_1
q_2/q_3	q_4	q_5/q_6
q_4	q_5/q_6	q_5/q_6
q_5/q_6	q_5/q_6	q_5/q_6
q_7	q_2/q_3	q_2/q_3

7. NFA to DFA [**Points**: 10]

For each of the following NFAs: describe the language that it accepts, convert it to DFA and remove unreachable states.

(a) $M_1 = (\Sigma, Q, \delta, q_{start}, F), \Sigma = \{0, 1\}, Q = \{q_1, q_2, q_3, q_4\}, q_{start} = q_1, F = \{q_4\}$

δ	0	1	ϵ
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\{\}$
q_2	$\{q_3\}$	$\{q_3\}$	$\{\}$
q_3	$\{q_4\}$	$\{q_4\}$	$\{\}$
q_4	$\{\}$	$\{\}$	$\{\}$

Solution:

$M_1' = (\Sigma, Q, \delta, q_{start}, F)$:
$\Sigma = \{0, 1\}$,
$Q = \{\{q_1\}, \{q_1, q_2\}, \{q_1, q_3\}, \{q_1, q_4\}, \{q_1, q_2, q_3\}, \{q_1, q_2, q_4\}, \{q_1, q_3, q_4\}, \{q_1, q_2, q_3, q_4\}\}$,
$q_{start} = \{q_1\}$,
$F = \{\{q_1, q_4\}, \{q_1, q_2, q_4\}, \{q_1, q_3, q_4\}, \{q_1, q_2, q_3, q_4\}\}$

δ	0	1
$\{q_1\}$	$\{q_1\}$	$\{q_1, q_2\}$
$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_1, q_3\}$	$\{q_1, q_4\}$	$\{q_1, q_2, q_4\}$
$\{q_1, q_4\}$	$\{q_1\}$	$\{q_1, q_2\}$
$\{q_1, q_2, q_3\}$	$\{q_1, q_3, q_4\}$	$\{q_1, q_2, q_3, q_4\}$
$\{q_1, q_2, q_4\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_1, q_3, q_4\}$	$\{q_1, q_4\}$	$\{q_1, q_2, q_4\}$
$\{q_1, q_2, q_3, q_4\}$	$\{q_1, q_3, q_4\}$	$\{q_1, q_2, q_3, q_4\}$

(b) $M_2 = (\Sigma, Q, \delta, q_{start}, F), \Sigma = \{a, b, c\}, Q = \{q_1, q_2, q_3, q_4, q_5\}, q_{start} = q_1, F = \{q_4\}$

δ	a	b	c	ϵ
q_1	$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_1\}$	$\{\}$
q_2	$\{\}$	$\{q_3\}$	$\{q_4\}$	$\{\}$
q_3	$\{\}$	$\{\}$	$\{q_5\}$	$\{\}$
q_4	$\{\}$	$\{\}$	$\{\}$	$\{q_1\}$
q_5	$\{\}$	$\{\}$	$\{\}$	$\{q_4\}$

Solution:

$M_2' = (\Sigma, Q, \delta, q_{start}, F)$:
$\Sigma = \{a, b, c\}$,
$Q = \{\{q_1\}, \{q_1, q_2\}, \{q_1, q_3\}, \{q_1, q_4\}, \{q_1, q_4, q_5\}\}$,
$q_{start} = \{q_1\}$,
$F = \{\{q_1, q_4\}, \{q_1, q_4, q_5\}\}$

δ	a	b	c
$\{q_1\}$	$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_1\}$
$\{q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_1, q_4\}$
$\{q_1, q_3\}$	$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_1, q_4, q_5\}$
$\{q_1, q_4\}$	$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_1\}$
$\{q_1, q_4, q_5\}$	$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_1\}$

Note: States $\{q_1, q_4\}$ and $\{q_1, q_4, q_5\}$ are equivalent and can be merged together.

8. DFA to regular expression [**Points**: 10]
Give a regular expression for each of the following DFAs:

(a) $M_1 = (\Sigma, Q, \delta, q_{start}, F), \Sigma = \{0, 1\}, Q = \{q_1, q_2, q_3, q_4, q_5\}, q_{start} = q_1, F = \{q_4\}$

δ	0	1
q_1	q_3	q_2
q_2	q_4	q_3
q_3	q_3	q_4
q_4	q_5	q_5
q_5	q_5	q_5

Solution:

$$10 \cup (0 \cup 11)0^*1$$

(b) $M_2 = (\Sigma, Q, \delta, q_{start}, F), \Sigma = \{0,1\}, Q = \{q_1, q_2, q_3, q_4, q_5\}, q_{start} = q_1, F = \{q_4, q_5\}$

δ	0	1
q_1	q_1	q_2
q_2	q_1	q_3
q_3	q_5	q_4
q_4	q_1	q_4
q_5	q_1	q_2

Solution:

$$0^*1((\epsilon \cup 10 \cup 11)0^*1)^*(11 + 10) \text{ or, equivalently, } (0 \cup 1)^*(111 \cup 110)$$

Note: use algorithm described in the class

9. Irregularity [**Points**: 10]
Let h be a homomorphism $h : \Sigma \to \Gamma^*$, we define **inverse homomorphism** as the following function: For any $w \in \Gamma^*$, let $h^{-1}(w) = \{s \in \Sigma^* \mid h(s) = w\}$ and $h^{-1}(B) = \{s \in \Sigma^* \mid h(s) \in B\}$ for any $B \subseteq \Gamma^*$.

Theorem1: The class of regular languages is closed under inverse homomorphism. That is, if B is regular then so is $h^{-1}(B)$.

Prove that the following languages are not regular using closure theorems, which were shown in the class:

(a) $B = \{0^n 1^m 0^k \mid 0 \leq n \leq m \leq k\}$

Solution:

Proof by contradiction. Let's assume that B is a regular language. If B is regular, then $A = \{1^m 0^k \mid 0 \leq m \leq k\} = B \cap 1^*0^*$ has to be regular by the closure theorem (since 1^*0^* is a regular expression that describes a regular language). Let's define a homomorphism h to be $h : \{0,1\} \to \{0,1\}^*$ with $h(0) = 1, h(1) = 0$, then $h(A) = \{0^m 1^k \mid 0 \leq m \leq k\}$ has to be regular. We also know that by the closure theorem $A^R = \{0^k 1^m \mid 0 \leq m \leq k\}$ has to be regular. $A^R \cap h(A)$ is a intersection of two regular languages and hence also a regular language (by another closure theorem). But $A^R \cap h(A) = \{0^k 1^m \mid 0 \leq m \leq k\} \cap \{0^m 1^k \mid 0 \leq m \leq k\} = \{0^n 1^n \mid 0 \leq n\}$ which is known to be not a regular language, contradiction. Hence, B is not a regular language.

(b) $C = \{a^n b a^n \mid n \geq 0\}$

Solution:

Proof by contradiction. Let's assume that C is a regular language. Let's define a homomorphism h to be $h : \{0, 1, b\} \to \{a, b\}^*$ with $h(0) = a, h(1) = a, h(b) = b$ then $h^{-1}(C) = \{(0 \cup 1)^n b(0 \cup 1)^n \mid n \geq 0\}$, which has to be regular by the Theorem1. 0^*b1^* is a regular language (since 0^*b1^* is a regular expression), hence, by the closure theorem, $h^{-1}(C) \cap 0^*b1^* = \{0^n b1^n \mid n \geq 0\}$ also has to be regular. Now let's define h' to be $h' : \{0, 1, b\} \to \{0, 1\}^*$ with $h(0) = 0, h(1) = 1, h(b) = \epsilon$. Since h' is a homomorphism, $h'(h^{-1}(C) \cap 0^*b1^*) = h'(\{0^n b1^n \mid n \geq 0\}) = \{0^n 1^n \mid n \geq 0\}$ has to be regular. But we know that $\{0^n 1^n \mid n \geq 0\}$ is not a regular language. We got a contradition, hence C is not a regular language.

Note: You may use the fact that $\{0^n 1^n \mid n \geq 0\}$, $\Sigma = \{0, 1\}$ is not regular without proving it. For part (b), you may find inverse homomorphism useful.