Problem Set 0 Solution

Fall 09

Due: Tuesday Sep 01 at 11:00 AM in class (i.e., Room 103 Talbot Lab)
Please follow the homework format guidelines posted on the class web page:
`http://www.cs.uiuc.edu/class/fa09/cs373/`

1. True/false/meaningless.
[**Category**: Notation, **Points**: 20]

   Answer each of the following with true, false or meaningless. Follow the notations in Sipser. We use $\{\ldots\}$ to represent sets (not, for example, multisets).

   D1) $\varnothing \in \{\varnothing, 1, 8\}$ **answer:** Y

   D2) $\varnothing \subseteq \{\varnothing, 1, 8\}$ **answer:** Y

   D3) $1 \subseteq \{\varnothing, 1, 8\}$ **answer:** N or Meaningless

   D4) $\{1\} + \{1, 2\} = \{1, 2\}$ **answer:** N or Meaningless

   D5) $\{1, 2\} \setminus \{1\} = \{2\}$ **answer:** Y

   D6) $\{1, 2\} \setminus \{3\} = \{0\}$ **answer:** N

   D7) $\{1, 2\} \cap \{3, 4\} = \{\}$ **answer:** Y

   D8) $\{1, 2\} \cap \{3, 4\} = \{\{\}\}$ **answer:** N

   D9) $\{1, 2\} \cap \{3, 4\} = \{\varnothing\}$ **answer:** N

   D10) $\{1, 2\} \cup \{1, 3\} = \{1, 1, 2, 3\}$ **answer:** Y

   D11) $\{1, 2\} \cup \{1, 3\} = \{1, 2, 3\}$ **answer:** Y

   D12) $\{1, \{1\}, \{\{1\}\}\} = \{1\}$ **answer:** N

   D13) $\{1, \{1\}, \{\{1\}\}\} = \{1, 1, 1\}$ **answer:** N

   D14) $\{1\} \in \{1, \{1\}, \{\{1\}\}\}$ **answer:** Y

   D15) $\{1\} \subseteq \{1, \{1\}, \{\{1\}\}\}$ **answer:** Y

   D16) $\{\{1\}\} \in \{1, \{1\}, \{\{1\}\}\}$ **answer:** Y

   D17) $\{a, b\} \times \{c, d\} = \{(a, b), (c, d)\}$ **answer:** N

   D18) $\Big(\{a, b\} \times \{c, d\}\Big) \cap \Big(\{c, d\} \times \{a, b\}\Big) = \{\}$ **answer:** Y

   D19) $|\{a, b, c\} \times \{d, e\}| = 6$ **answer:** Y

   D20) $\{a, b\} \times \{\} = \{a, b\}$ **answer:** N

   D21) $\{a, b\} \setminus \{b, a\} = \{\}$ **answer:** Y

   For grading, since we have 20 points and 21 questions, we have 1 point each for each correct answer up to 20 points total (that is, first error is free).

2. Divisible by 9.
[**Category**: Proof, **Points**: 20]

Prove that for any positive integer n, the quantity $4^n + 15n - 1$ is divisible by 9. We recommend to use induction. Note that you may need to apply induction more than once.

PROOF. We prove via induction. The questions asks for all $n > 0$, therefore the base case is $n = 1$. Define $a_n := 4^n + 15n - 1$.

(base case). $a_1 = 4^1 + 15 * 1 - 1 = 18$. Hence 9 divides a_1.

(inductive case). Assuming 9 divides a_k, we need to show 9 divides a_{k+1}. This is equivalent to showing that 9 divides $a_{k+1} - a_k = 3 * 4^k + 15$. To show that 9 divides $b_k = 3 * 4^k + 15, k \geq 1$, we use another induction:

(base case). $b_1 = 3 * 4^1 + 15 = 63$. 9 divides 63.

(inductive case). Assuming 9 divides b_i, we need to show 9 divides b_{i+1}. This is equivalent to showing that 9 divides $b_{i+1} - b_i = 3 * 4^{i+1} - 3 * 4^i = 9 * 4^i$. We are done. □

For grading, each base case is worth 3 points and each inductive case is worth 7 points. It is also possible to use a single induction:

SECOND PROOF. We prove via induction. The questions asks for all $n > 0$, therefore the base case is $n = 1$. Define $a_n := 4^n + 15n - 1$.

(base case). $a_1 = 4^1 + 15 \cdot 1 - 1 = 18$. Hence 9 divides a_1.

(inductive case). Assuming 9 divides a_k, we need to show 9 divides a_{k+1}. $a_{k+1} = 4^{k+1} + 15(k+1) - 1 = 4^k * 4 + (60k - 45k) + (-4 + 18) = 4(4^k + 15k - 1) - 45k + 18 = 4(4^k + 15k - 1) + 9(-5k + 2) = 4a_k + 9(-5k + 2)$. We are done. □

For grading, the base case is worth 3 points and the inductive case is worth 17 points. There are probably other ways to do the induction and points should be given accordingly.

3. Countably Infinite.
[**Category**: Proof, **Points**: 20]

Prove that the set of all valid Java programs is countably infinite.
Hint: Try to build an injection from this set into the natural numbers.

PROOF. A java program must load into memory and therefore is a finite string of java code. We need to show that the set of finite java code strings maps to an infinite subset of natural numbers. This set is obviously infinite since one can have longer and longer strings of valid java code. We are left to show that a string of java code can be uniquely mapped to an integer (injective, or one-to-one). We obtain one such mapping by simply viewing a java code string as hex string, which corresponds to a unique integer. The mapping is injective by construction since if java code strings s_1 and s_2 map to the same integer n, they must be the same if viewed as hex strings and therefore, the same java program. $\square$

For grading, it is important to show the following pieces:

- The set of java programs is an infinite set (6 points).
- Give a mapping (describing one is fine) from a java program to an integer (6 points).
- Show that the mapping is injective (8 points).

4. Coloring.

[**Category**: Proof, **Points**: 20] There are n lines on the plain such that no three of them pass through a common point. They partition the plain into a number of cells (each cell is a connected piece of plain bounded to some of these lines). Prove that we can color these cells "Red" and "Blue" so that no two cells whose boundary share at least two points are in the same color (Again we recommend using induction).

PROOF.

(base case). When there is $n = 1$ line, the plain is split into two parts. We can simply color one part red, one part blue. It is also possible start with $n = 0$, in which case a single color will do (for grading: 6 points).

(inductive case). Assume the statement holds for $n = k$. For $n = k + 1$, if we remove any line l, we have k lines left, which we can color red and blue correctly by assumption (first figure). Adding back the line l, some new cells are not colored correctly, which we need to fix (C_1, C_2, C_4, C_6 in the second figure). We perform the fix by simply flipping the colors of all cells lying on one side (say the upper right side) of the line l (third figure). We now have a valid coloring because 1) For cells on the lower left side of l, the coloring is valid by assumption; 2) For cells bordering l, the flipping resolves any conflicts caused by adding line l; and 3) For cells on the upper right side of l, the coloring was valid and flipping the colors of all cells preserves that property (for grading: 14 points). $\square$