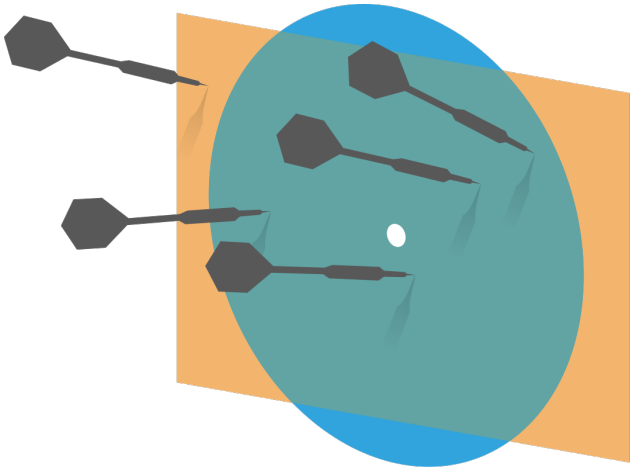
Probability and Statistics for Computer Science



Credit: wikipedia

"Its straightforward to link a number to the outcome of an experiment. The result is a **Random variable**." ---Prof. Forsythe

Random variable is a function, it is not the same as in **X = X+1**

Last time

- Conditional Probability
  - Review the main concepts
  - *Examples and Misconceptions* of Conditional Probability
  - Independence

Motivation of the use of random variable

- Amount of money put on a bet
- Age at retirement of a population
- Rate of vehicles passing by the toll
- SAT scores of students in a college

The random event that hasn’t been mapped to numbers

Content

- Random Variable
  - **What is Random Variable?**
  - *Probability distribution* of random variable
  - *Joint & Conditional probability distribution* of random variable

Random variables

- ***Every outcome will have an assigned number*** *using the function* $X(\omega)$*,* ω *is an outcome.*
- *Because random variable is associated with event(s), it carries the* ***randomness*** *due to the associated event(s).*

Random variables

- A random variable can have a defined range of values through the mapping from events.

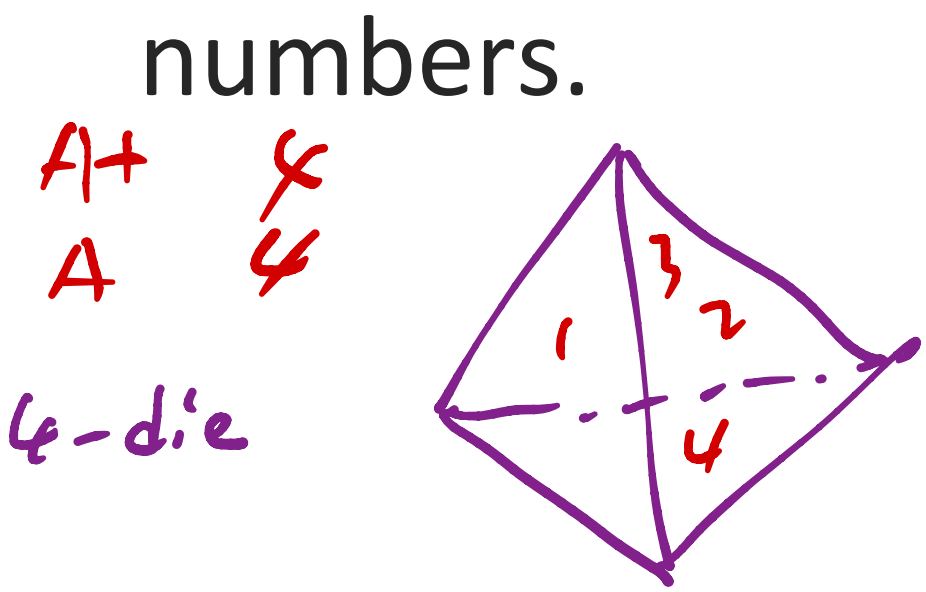
$X(\omega) = \begin{cases} 0 & \text{tail} \\ 1 & \text{head} \end{cases}$

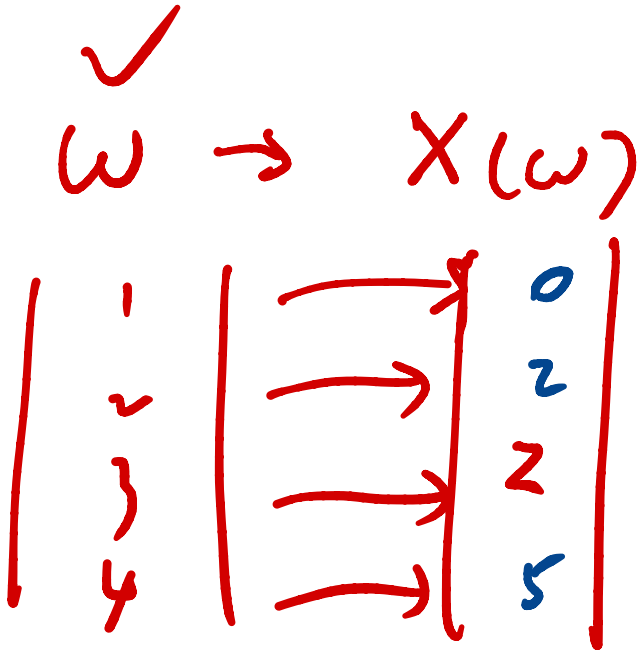
Random variables

- The values of a random variable can be either **discrete**, **continuous** or **mixed**.

Discrete Random variables

- The range of a discrete random variable is a countable set of real numbers.





Random Variable Example

- **Number of pairs in a hand of 5 cards**
  - Let a single outcome be the hand of 5 cards
  - Each outcome maps to values in the set of numbers {0, 1, 2}

Random Variable Example

- **Number of pairs in a hand of 6 cards**
  - Let a single outcome be the hand of 6 cards
  - What is the range of values of this random variable?

{0, 1, 2, 3}

Random Variable Example

- **Number of pairs in a hand of 6 cards**
  - Let a single outcome be the hand of 6 cards
  - What is the range of values of this random variable?

{0, 1, 2, 3}

Q: Random Variable

- If we roll a 3-sided die, and define random variable X *so that* X takes on -1, 0, and 1 respectively for outcomes showing side 1, 2 and 3. Then what is the range, the set of numbers that X^2 takes on?

A. {-1, 0, 1}

B. {0, 1}

Q: Random Variable

- If we roll a 3-sided die, and define random variable X *so that* X takes on -1, 0, and 1 respectively for outcomes showing side 1, 2 and 3. Then what is the range, the set of numbers that X^2 takes on?

A. {-1, 0, 1}

B. {0, 1}

Three important facts of Random variables

- Random variables have **probability functions**
- Random variables can be **conditioned** on events or other random variables
- Random variables have **averages**

Random variables have **probability functions**

- Let X be a random variable & A is the event X *is mapping*
- The set of outcomes $\{A \in \Omega \mid X(A) = x\}$ is an event with probability

$$P(X = x)$$

X is the random variable

x is any unique instance that X takes on

Random variable: die example

Roll 4-sided fair die twice.

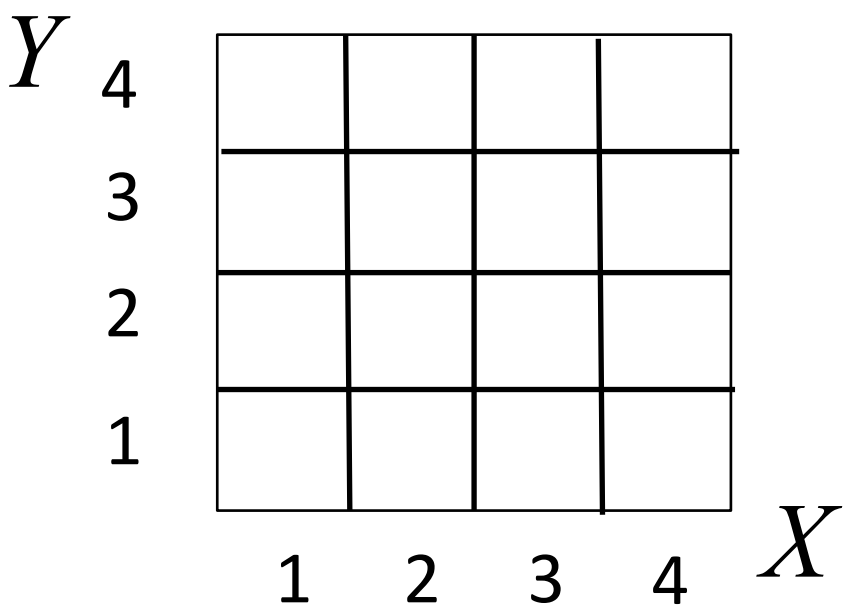
Define these random variables:

X, the values of 1st roll

Y, the values of 2nd roll

Sum $S = X + Y$

Difference $D = X - Y$



4×4=16

Size of Sample Space = ?

Random variable: die example

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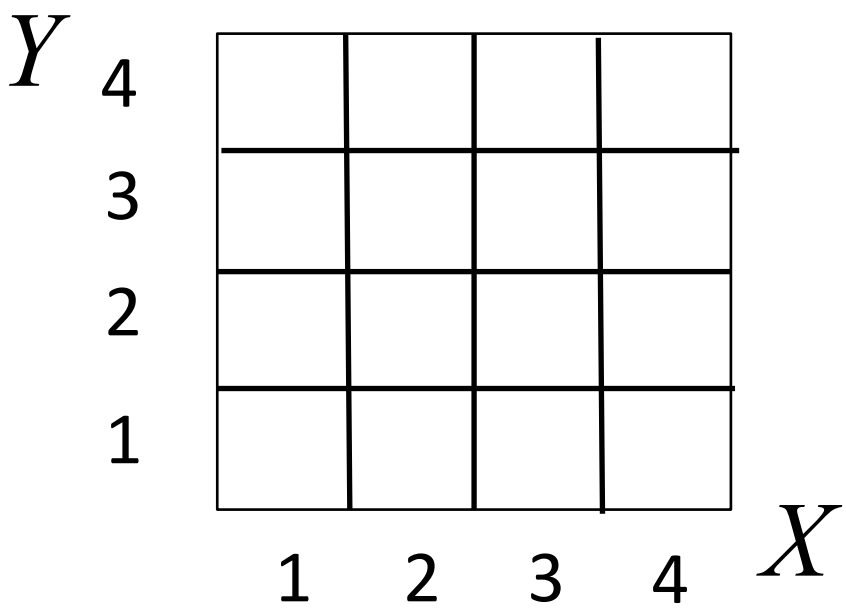


Size of Sample Space = 16

Random variable: die example

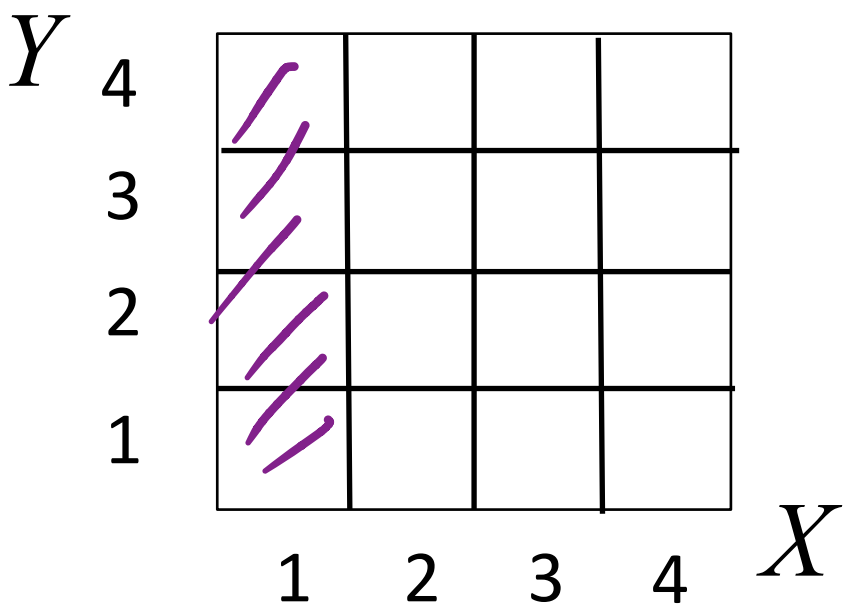
Roll 4-sided fair die twice.

$P(X = 1) = \frac{4}{16} = \frac{1}{4}$

$P(Y \leq 2) = \frac{8}{16} = \frac{1}{2}$

$P(S = 7)$

$P(D \leq -1)$



Size of Sample Space = 16

Random variable: die example

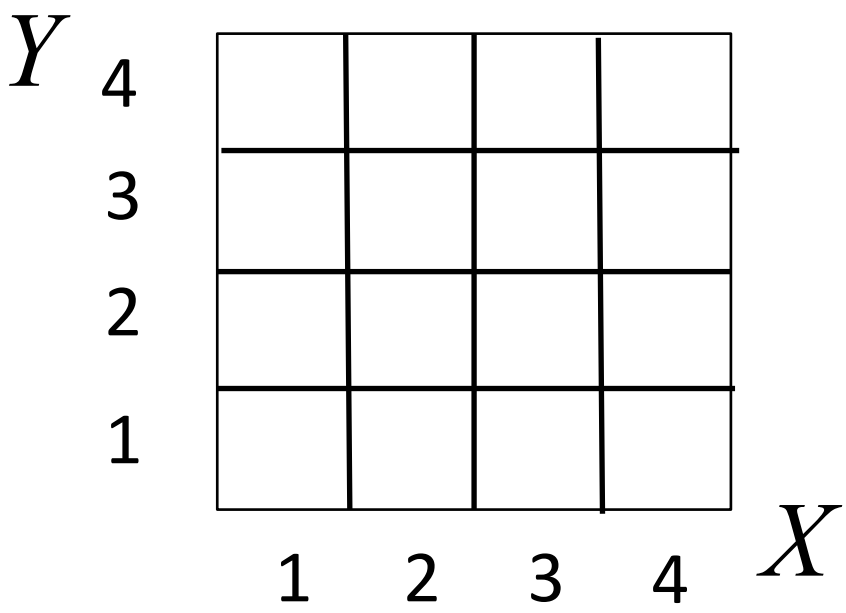
Roll 4-sided fair die twice.

$P(X = 1)$ $\frac{1}{4}$

$P(Y \leq 2)$ $\frac{1}{2}$

$P(S = 7)$

$P(D \leq -1)$



Size of Sample Space = 16

Random variable: die example

$S = X + Y$

Y \ X	1	2	3	4
4	5	6	7	8
3	4	5	6	7
2	3	4	5	6
1	2	3	4	5

$D = X - Y$

Y \ X	1	2	3	4
4	-3	-2	-1	0
3	-2	-1	0	1
2	-1	0	1	2
1	0	1	2	3

$$P(S = 7) = \frac{2}{16} = \frac{1}{8}$$

$$P(D \leq -1) = \frac{6}{16} = \frac{3}{8}$$

Random variable as vectors

Brain imaging of Human emotions

A) Moral conflict

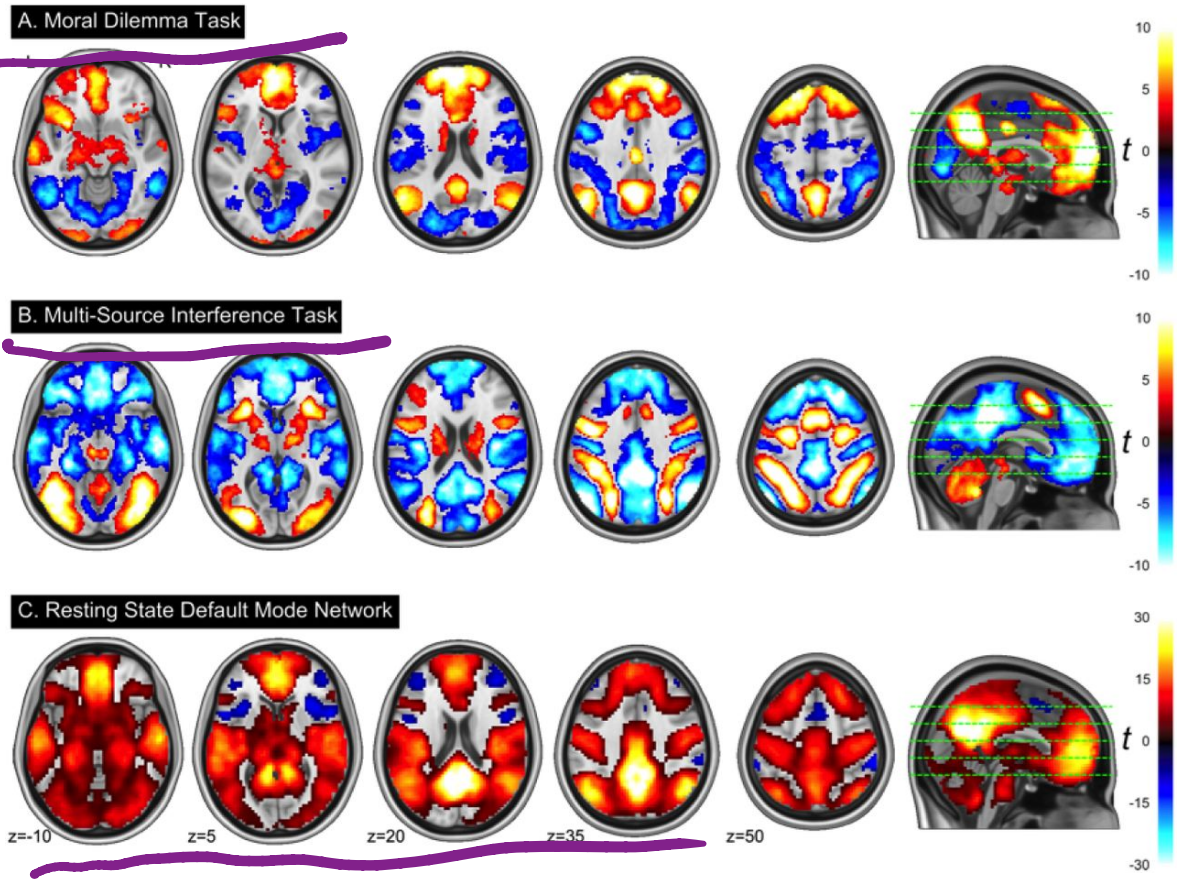
B) Multi-task

C) Rest

$X(A)$

$\downarrow$

$\{s_A, x_A, y_A, z_A, t_A\}$



A. McDonald et al. NeuroImage doi: 10.1016/j.neuroimage.2016.10.048

Content

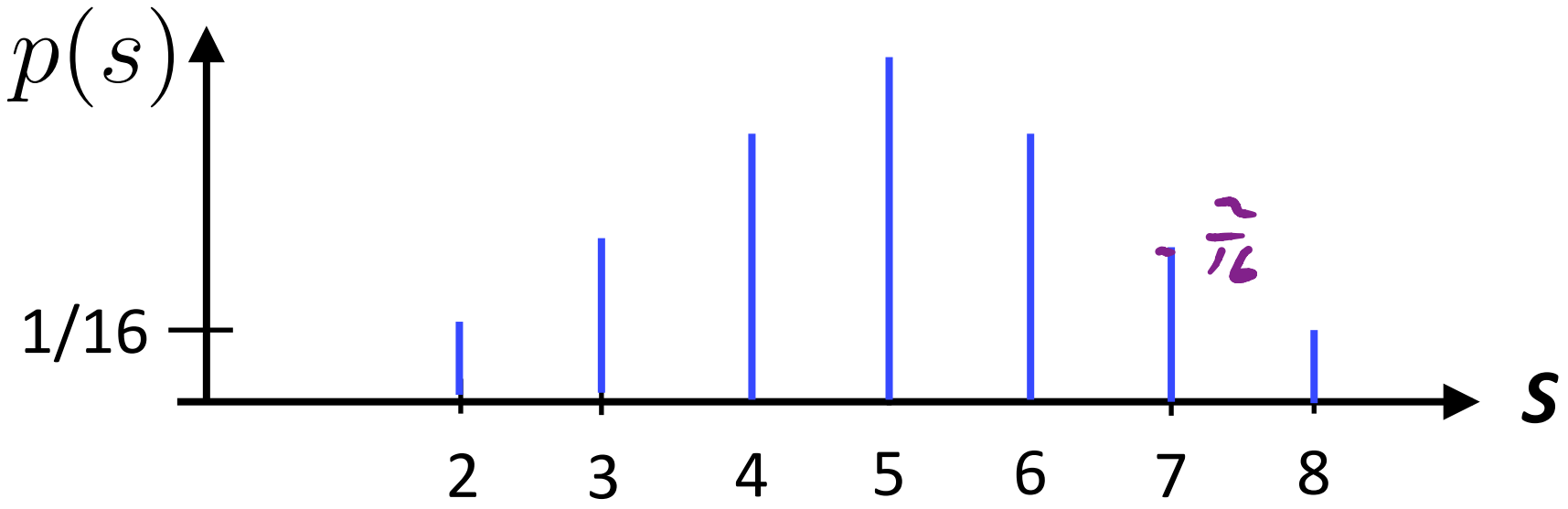
- Random Variable
  - The Definition of Random Variable
  - ***Probability distribution* of random variable**
  - *Joint & Conditional probability distribution* of random variable

Random variables have **probability functions**

- $P(X = x)$ is called the probability distribution for all possible $\boldsymbol{x}$
- $P(X = x)$ is also denoted as $P(x)$ or $p(x)$
- $P(X = x) \geq 0$ for all values that X can take, and is 0 everywhere else
- The sum of the probability distribution is 1 $\sum_x P(x) = 1$

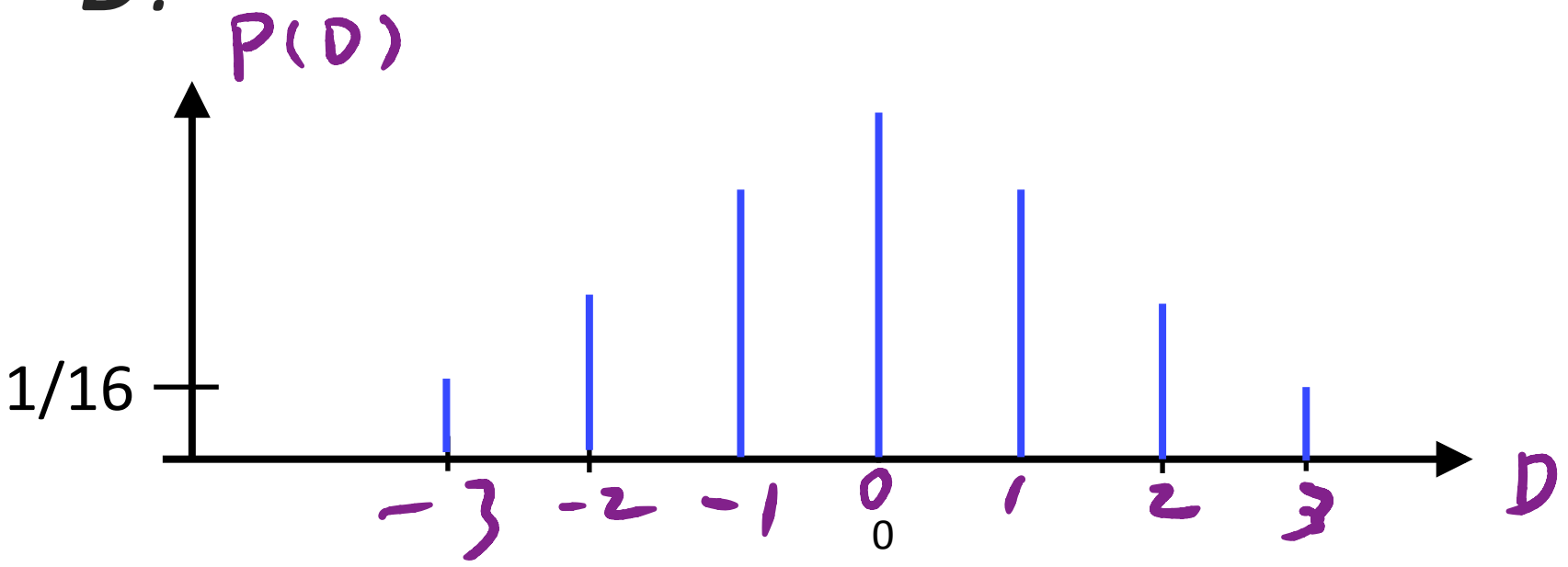
Probability distribution

- Give the random variable ***S*** *in the 4-sided die, whose range is* {2,3,4,5,6,7,8}, probability distribution of ***S.***



Probability distribution

- Give the random variable ***D = X-Y,*** *what is the probability distribution of* ***D****?*



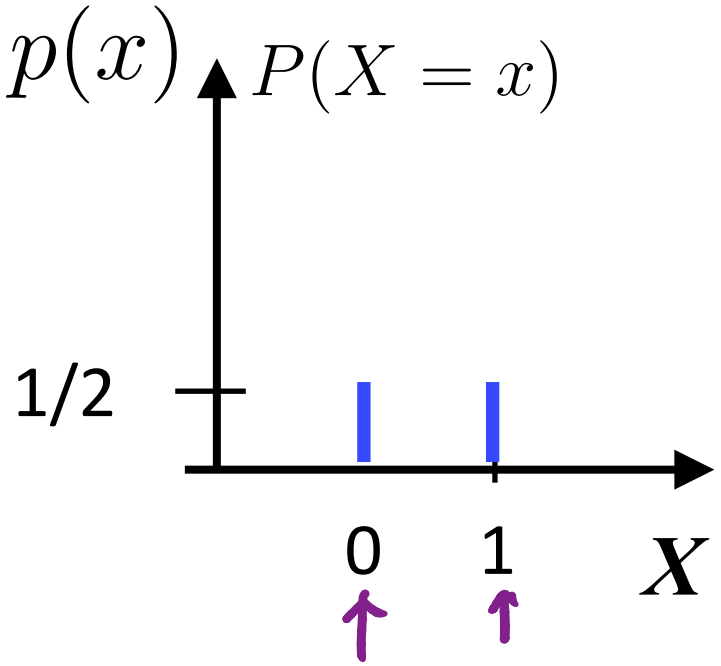
Cumulative distribution

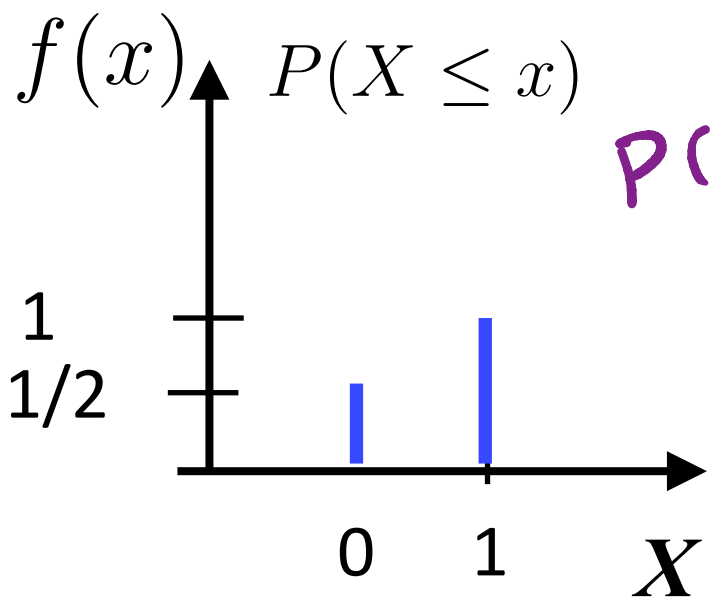
- $P(X \leq x)$ is called the cumulative distribution function of X
- $P(X \leq x)$ is also denoted as $f(x)$
- $P(X \leq x)$ is a non-decreasing function of x

Probability distribution and cumulative distribution

- Give the random variable ***X***,

$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$





P(x≤1) = 1

Q. What is the value?

A biased four-sided die is rolled once. Random variable X is defined to be the down-face value.

$$P(X = X_0) = \begin{cases} \frac{X_0}{10} & X_0 = 1, 2, 3, 4 \\ 0 & \text{all others} \end{cases}$$

$P(X \leq 4)$

A) 0.1

B) 0.3

C) 0.2

D) 0.6

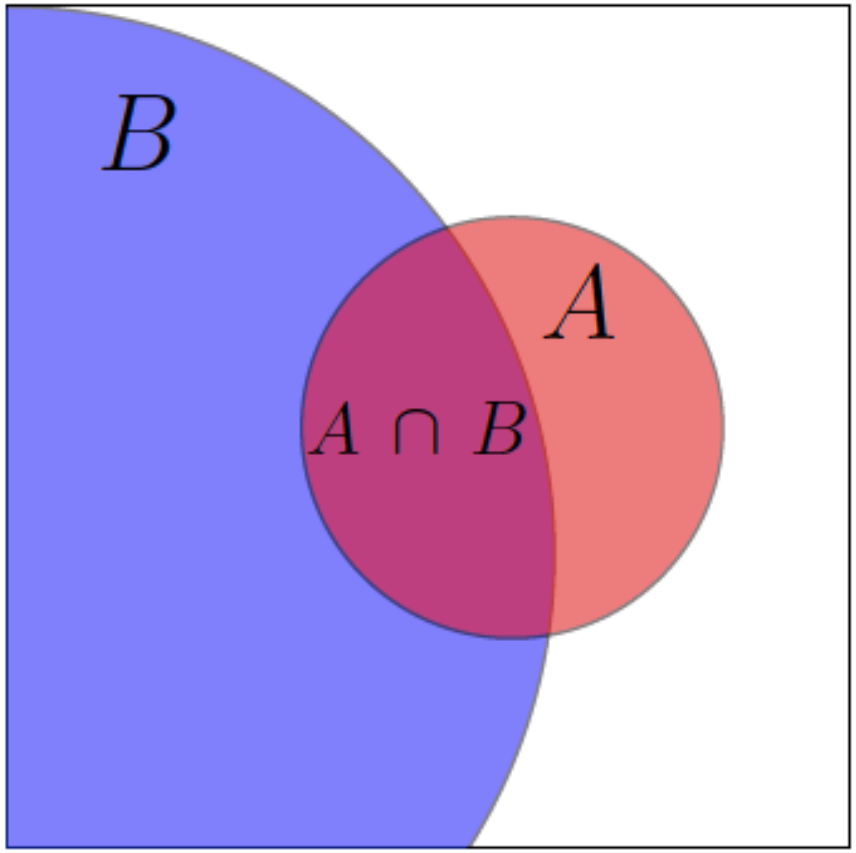
E) 1

Content

- Random Variable
  - The Definition of Random Variable
  - *Probability distribution* of random variable
  - ***Joint & Conditional probability distribution* of random variable**

Conditional Probability

- The probability of ***A*** given ***B***



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \neq 0$$

The "Size" analogy

Credit: Prof. Jeremy Orloff & Jonathan Bloom

Joint probability distribution (I)

- The purpose is to study relationship
- The joint probability distribution of two random variables ***X*** and ***Y*** is

$P(\{X = x\} \cap \{Y = y\})$ also denoted $P(x, y)$ for short.

$p(x, y)$ $p(x, y)$

Joint probability distribution (II)

- We can recover the individual probability distributions from the joint probability distribution

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

$p(x,y) = p(y|x)p(x)$

$\sum_y p(x,y)$

$= \sum_y p(y|x)\, p(x)$

$= p(x)\left(\sum_y p(y|x)\right)$, $\sum_y p(y|x) = 1$

Joint probability distribution (III)

- The sum of the joint probability distribution

$$\sum_{y}\sum_{x} P(x,y) = 1$$

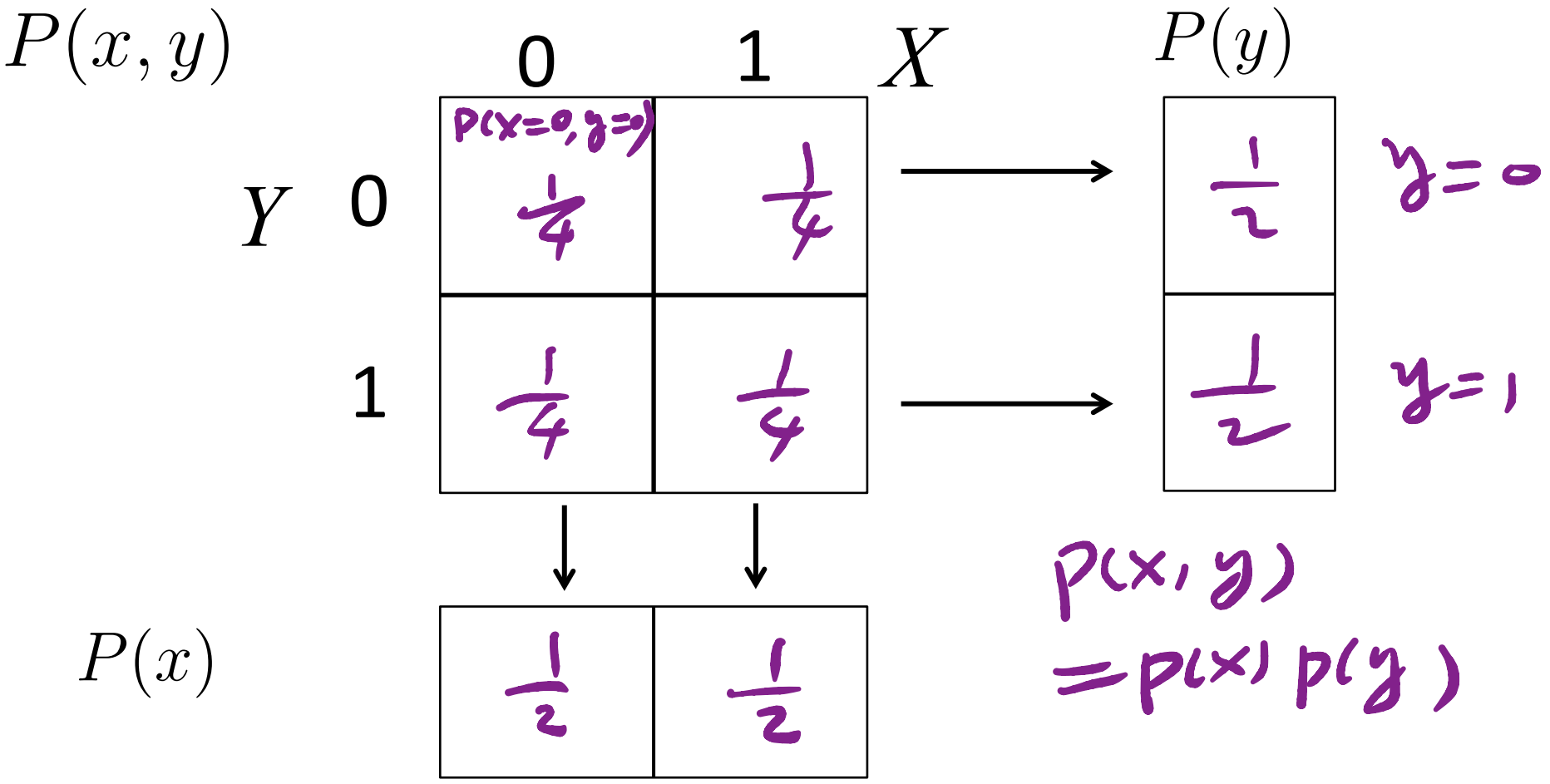
Joint Probability Example

- Tossing a coin twice, we define random variable $X\ and\ Y$ for each toss.

$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

Joint probability distribution example



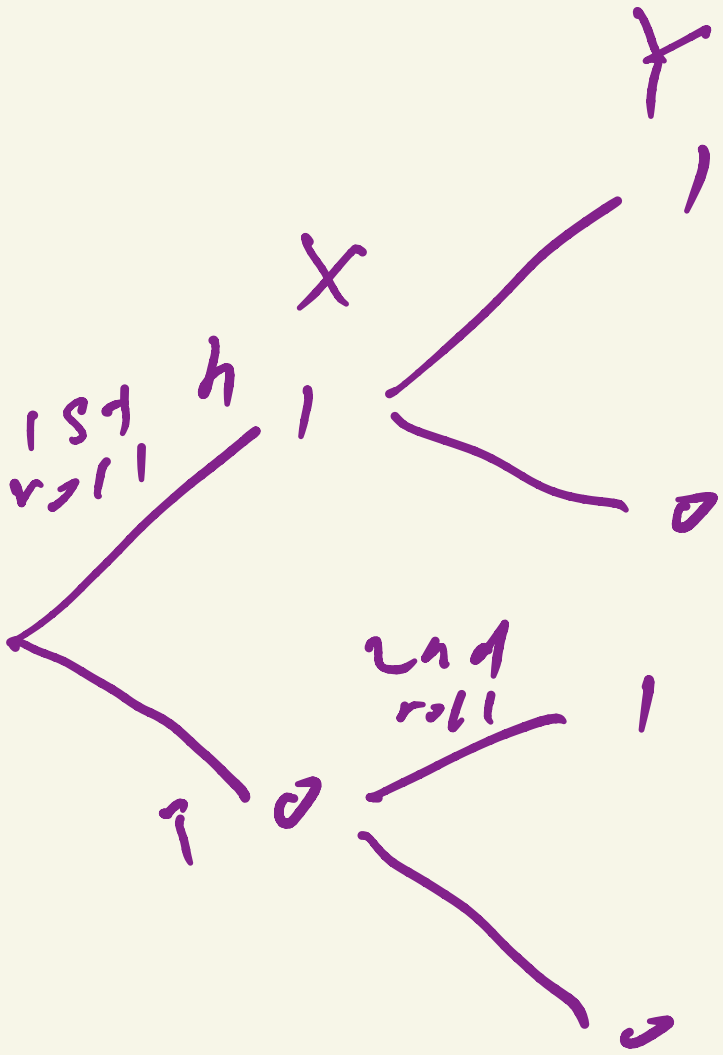
Joint Probability Example

Now we define Sum ***S*** = $X + Y$,

Difference ***D*** = $X - Y$. ***S*** takes on values {0,1,2} and ***D*** takes on values {-1, 0, 1}

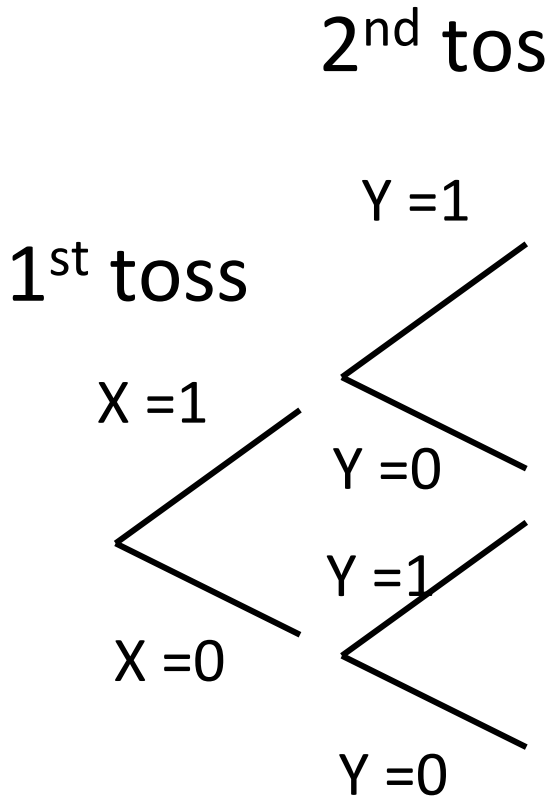
$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$



$S = X+Y$	$D = X-Y$	$P(S,D)$
2	0	$P(X=1)\,P(Y=1) = \frac{1}{4}$
1	1	$\frac{1}{4}$
1	-1	$\frac{1}{4}$
0	0	$\frac{1}{4}$

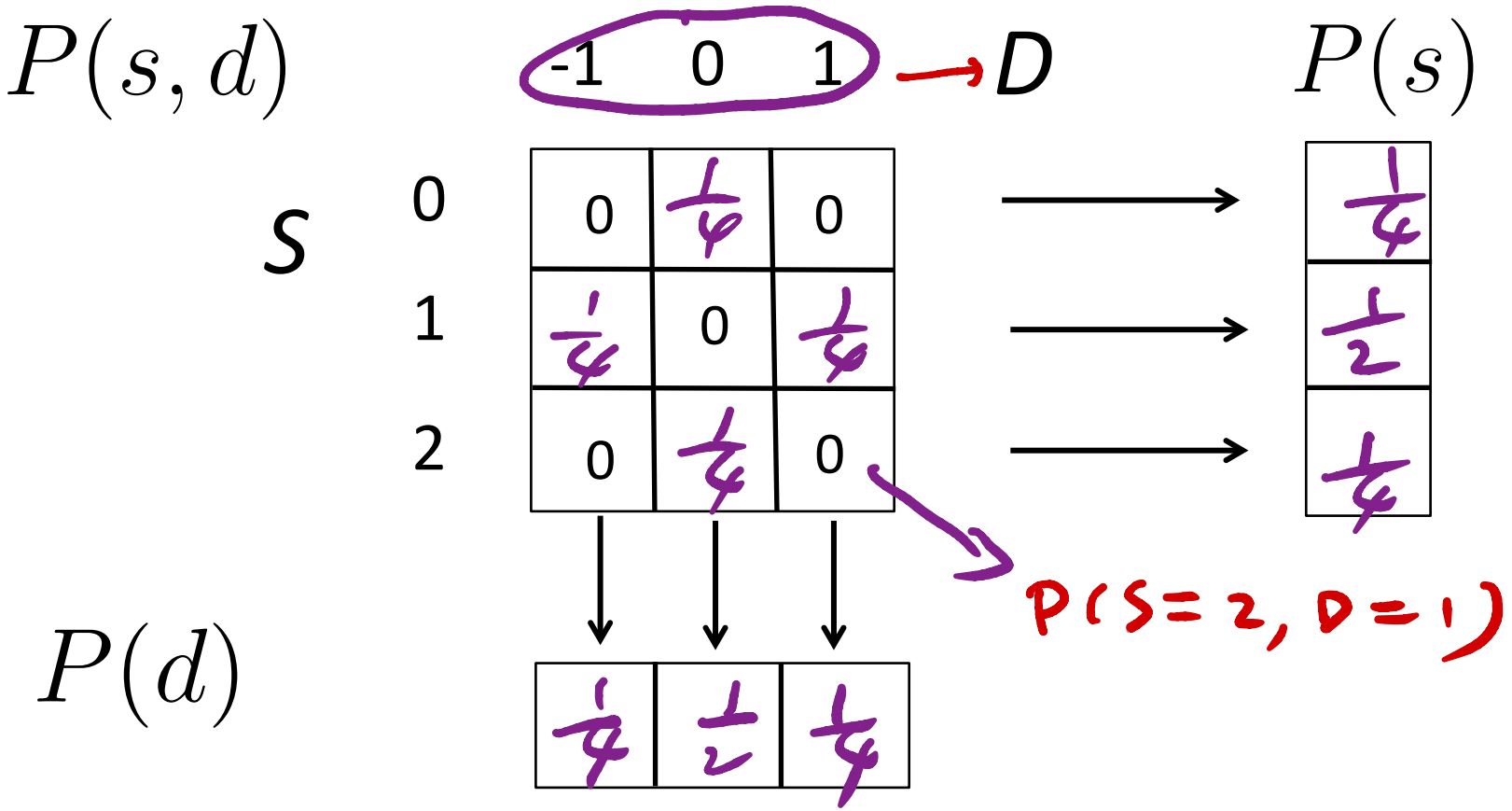
Joint Probability Example



S	D	P(s, d)
2	0	$\frac{1}{4}$
1	1	$\frac{1}{4}$
1	-1	$\frac{1}{4}$
0	0	$\frac{1}{4}$

Suppose coin is fair, and the tosses are independent

Joint probability distribution example



Independence of random variables

- Random variable X *and* Y are independent if

$$P(x, y) = P(x)P(y) \quad for\ all\ x\ and\ y$$

- In the previous coin toss example
  - Are X and Y independent?
  - Are ***S*** and ***D*** independent?

Independence of random variables

- Random variable X *and* Y are independent if

$$P(x, y) = P(x)P(y) \quad for\ all\ x\ and\ y$$

- In the previous coin toss example
  - X and Y are independent
  - ***S*** and ***D*** are not independent

Joint probability distribution example

$P(x, y)$

Y \ X	0	1	$P(y)$
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$P(x)$	$\frac{1}{2}$	$\frac{1}{2}$	

$p(x, y) = p(x)\,p(y)$

X, Y are indpt.

Joint probability distribution example

$P(s,d)$

S \ D	-1	0	1
0	0	$\frac{1}{4}$	0
1	$\frac{1}{4}$	0	$\frac{1}{4}$
2	0	$\frac{1}{4}$	0

$P(s)$

$\frac{1}{4}$
$\frac{1}{2}$
$\frac{1}{4}$

$P(d)$

$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

S & D Not indpt!!

$P(S=1, D=0) \neq P(S=1)P(D=0)$

$P(S=1, D=0) = 0$

$P(S=1)P(D=0)$

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Conditional probability distribution of random variables

- The conditional probability distribution of X *given* Y is

$$P(x|y) = \frac{P(x,y)}{P(y)} \quad P(y) \neq 0$$

- For any given Y, $\sum_x P(x|y) = 1$

- If X and Y are independent, all (x,y)

$$P(x,y) = P(x)P(y),$$
$$so\ P(x|y) = P(x)$$

Joint probability distribution example

$$P(s|d) = \frac{P(s,d)}{P(d)}$$

S \ D	-1	0	1
0	0	$\frac{1}{2}$	0
1	1	0	1
2	0	$\frac{1}{2}$	0

$P(s)$

$\frac{1}{4}$
$\frac{1}{2}$
$\frac{1}{4}$

$P(d)$

$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

P(S=1|d=-1)

p(d=-1|s=1) = ?

Bayes rule for random variable

- Bayes rule for events generalizes to random variables

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

$P(X=x \mid Y=y)$

$$= \frac{P(y|x)P(x)}{\sum_x P(y|x)P(x)}$$

Total Probability

Joint probability distribution example

$$P(s|d) = \frac{P(s,d)}{P(d)}$$

S \ D	-1	0	1	$P(s)$
0	0	$\frac{1}{2}$	0	$\frac{1}{4}$
1	1	0	1	$\frac{1}{2}$
2	0	$\frac{1}{2}$	0	$\frac{1}{4}$

$P(d)$			
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$P(D=-1|S=1)=?$

~~Joint~~ Cond; probability distribution example

$$P(s|d) = \frac{P(s,d)}{P(d)}$$

S \ D	-1	0	1	$P(s)$
0	0	$\frac{1}{2}$	0	$\frac{1}{4}$
1	1	0	1	$\frac{1}{2}$
2	0	$\frac{1}{2}$	0	$\frac{1}{4}$

$P(d)$			
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$P(D=-1|S=1) = \frac{P(S=1|D=-1)P(D=-1)}{P(S=1)} = \frac{1 \times \frac{1}{4}}{\frac{1}{2}}$$

$= \frac{1}{2}$

Assignments

- Chapter 4 of the textbook
- Next time: More random variable, Expectations, Variance

* Multiple choice questions in discussion.
  1 bonus point if you answer all Qs right.

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

See you next time

See You!