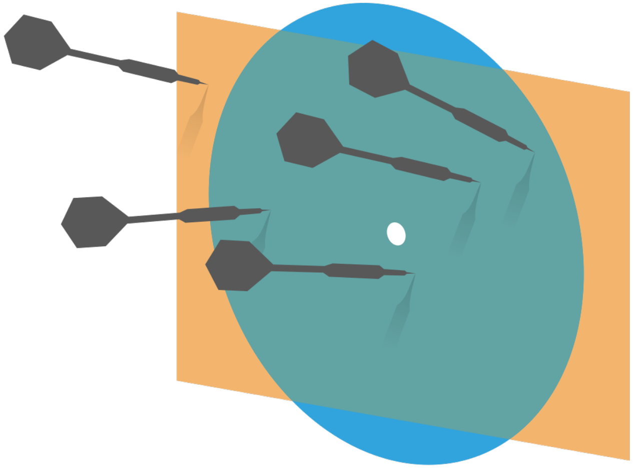
Probability and Statistics for Computer Science



Credit: wikipedia

"A major use of probability in statistical inference is the updating of probabilities when certain events are observed" – Prof. M.H. DeGroot

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 9.8.2021

Laws of Sets

Commutative Laws

$A \cap B = B \cap A$
$A \cup B = B \cup A$

Associative Laws

$(A \cap B) \cap C = A \cap (B \cap C)$
$(A \cup B) \cup C = A \cup (B \cup C)$

Distributive Laws

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Laws of Sets

Idempotent Laws

$A \cap A = A$
$A \cup A = A$

Identity Laws

$A \cup \emptyset = A$
$A \cap U = A$
$A \cup U = U$
$A \cap \emptyset = \emptyset$

Involution Law $(A^c)^c = A$

Complement Laws

$A \cup A^c = U$
$A \cap A^c = \emptyset$
$U^c = \emptyset$
$\emptyset^c = U$

De Morgan's Laws

$(A \cap B)^c = A^c \cup B^c$
$(A \cup B)^c = A^c \cap B^c$

U is the complete set

Objectives

- Probability
  - More probability calculation
  - Conditional Probability
    - Bayes rule
    - Independence

Senate Committee problem

The United States Senate contains **two** senators from each of the **50** states. If a committee of eight senators is selected at random, what is the probability that it will contain at least one of the two senators from **IL**?

Probability: Birthday problem

- Among 30 people, what is the probability that at least 2 of them celebrate their birthday on the same day? Assume that there is no February 29 and each day of the year is equally likely to be a birthday.

Conditional Probability

- Motivation of conditional probability

Conditional Probability

- Example:

An insurance company knows in a population of 100 thousands females, 89.835% expect to live to age 60, while 57.062% can expect to live to 80. Given a woman at the age of 60, what is the probability that she lives to 80?

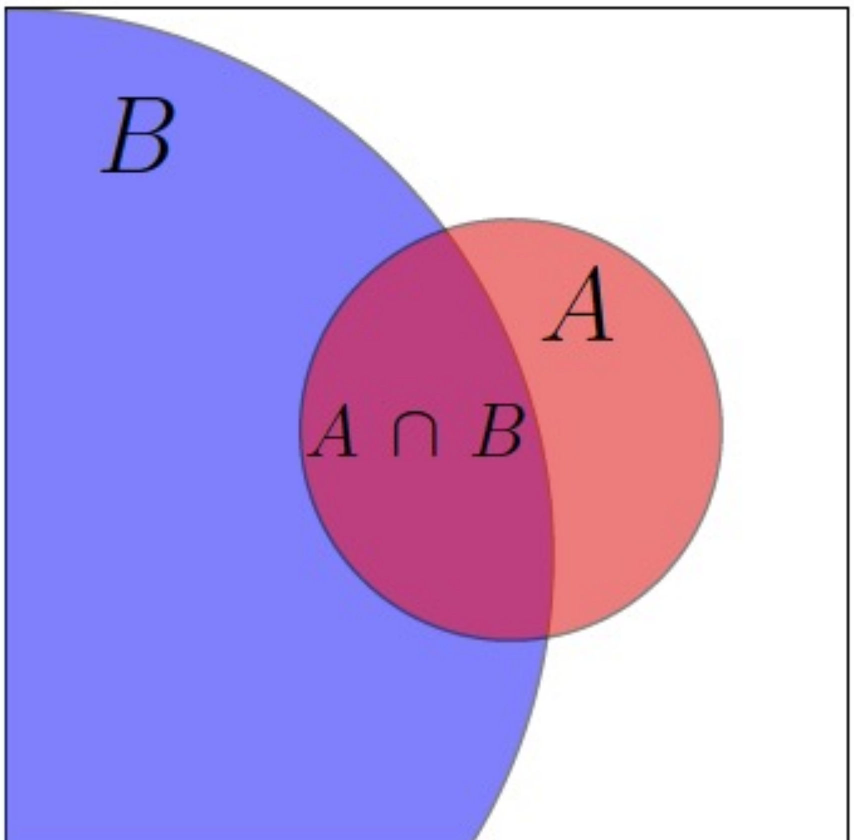
Conditional Probability

- Given the condition she is 60 already, the size of the sample space for the outcomes has changed to

  89,835 instead of 100,000

Conditional Probability

- The probability of ***A*** given ***B***



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \neq 0$$

The "Size" analogy

Credit: Prof. Jeremy Orloff & Jonathan Bloom

Conditional Probability

A : a woman lives to 80

$$P(A|B) = \frac{57,062}{89,835} = 0.6352$$

B : a woman is at 60 now

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

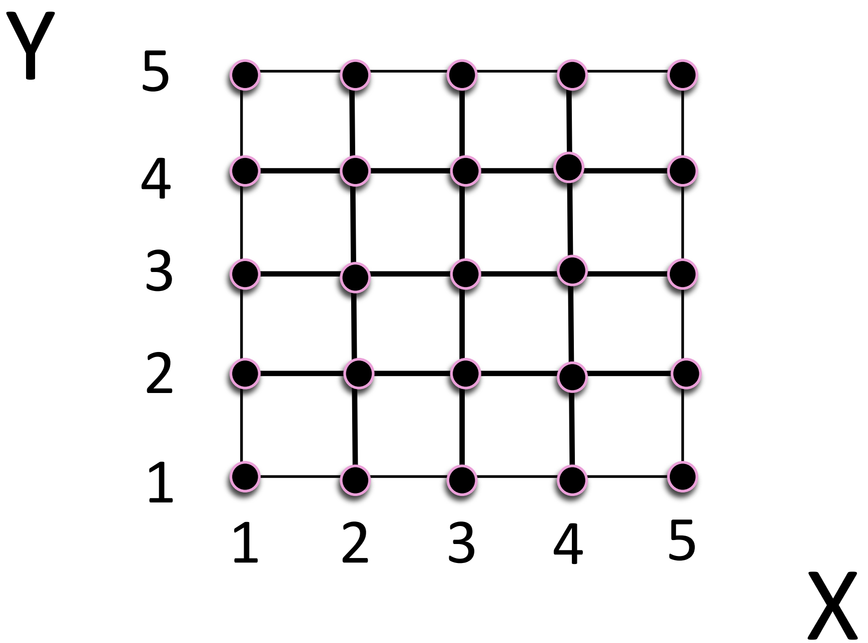
While $P(A) = \frac{57,062}{100,000} = 0.57062$

Conditional Probability: die example

Throw 5-sided fair die twice.

$$A : max(X, Y) = 4$$
$$B : min(X, Y) = 2$$

$$P(A|B) = ?$$



Conditional probability, that is?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

Multiplication rule using conditional probability

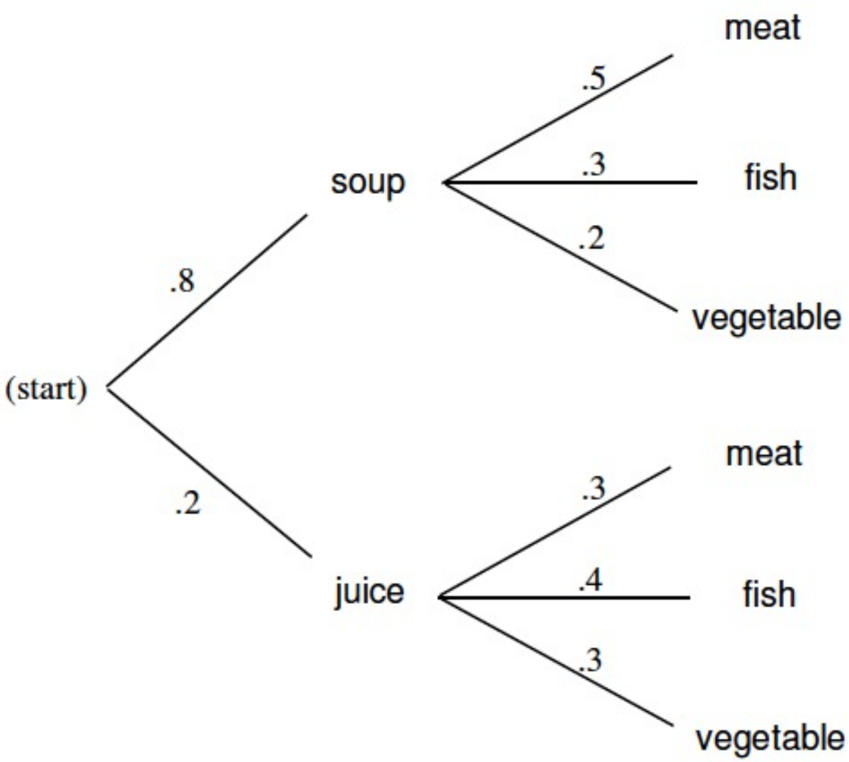
- Joint event

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

$$\Rightarrow \boxed{P(A \cap B) = P(A|B)P(B)}$$

Multiplication using conditional probability

$$P(A \cap B) = P(A|B)P(B)$$



$$P(soup \cap meat) =$$
$$P(meat|soup)P(soup)$$
$$= 0.5 \times 0.8 = 0.4$$

Symmetry of joint event in terms of conditional prob.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$

$$\Rightarrow P(B \cap A) = P(B|A)P(A)$$

Symmetry of joint event in terms of conditional prob.

$$\because P(B \cap A) = P(A \cap B)$$

$$\Rightarrow$$

$$P(A|B)P(B) = P(B|A)P(A)$$

The famous **Bayes** rule

$$P(A|B)P(B) = P(B|A)P(A)$$

$$\Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Thomas Bayes (1701-1761)

Bayes rule: lemon cars

There are two car factories, **A** and **B**, that supply the same dealer. Factory **A** produced **1000** cars, of which **10** were lemons. Factory **B** produced **2** cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory **B**?

Bayes rule: lemon cars

There are two car factories, A and B, that supply the same dealer. Factory A produced 1000 cars, of which 10 were lemons. Factory B produced 2 cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory B?

$$P(B|L) = \frac{P(L|B)P(B)}{P(L)}$$

Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

$$P(A|L) = ?$$

Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

$$P(A|L) = ?$$

$$P(A|L) = \frac{P(L|A)P(A)}{P(L)}$$

Or in this case

$$P(A|L) = 1 - P(B|L)$$

Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

$$P(A|L) = ?$$

$$P(A|L) = \frac{P(L|A)P(A)}{P(L)}$$

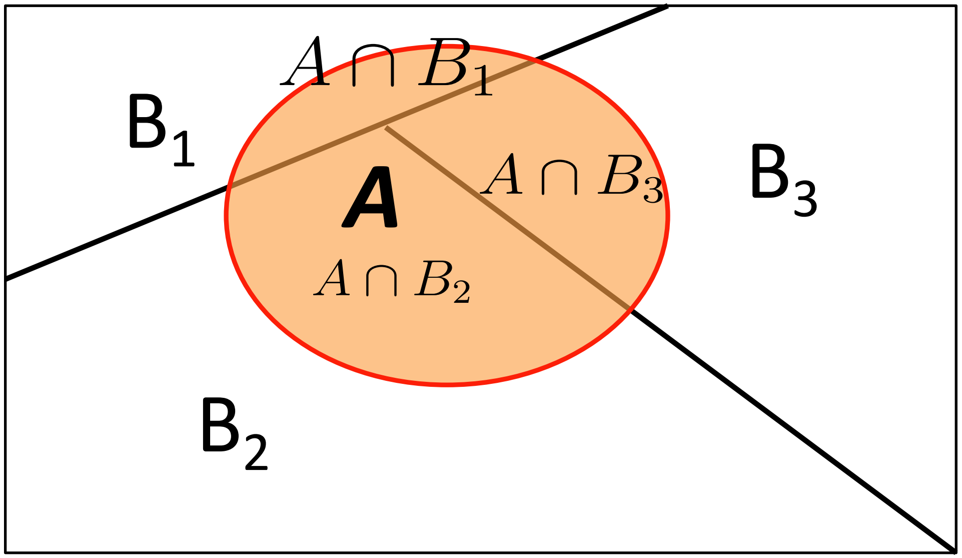
Or in this case

$$P(A|L) = 1 - P(B|L)$$

=

Total probability

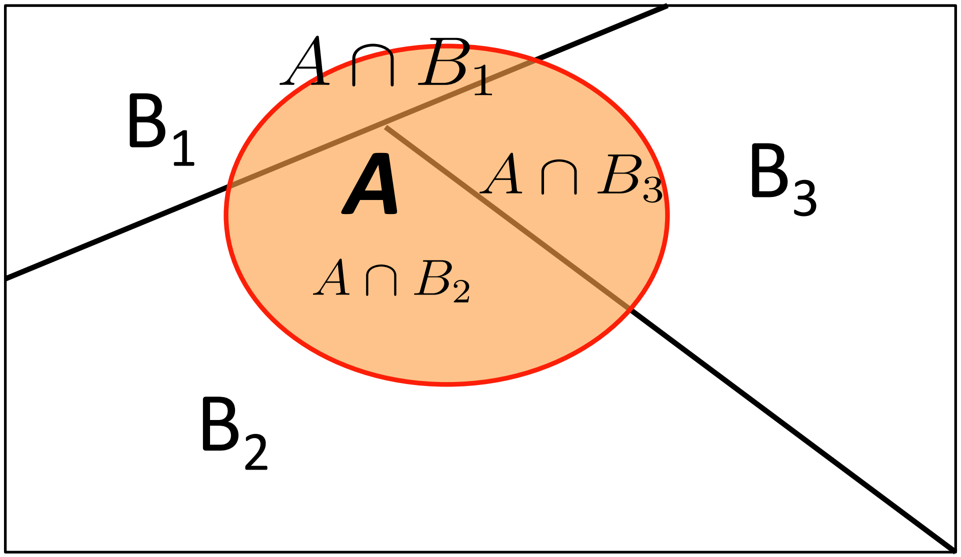
$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$
$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$



Total probability general form

$$P(A) = \sum_j \left(P(A|B_j)P(B_j)\right)$$

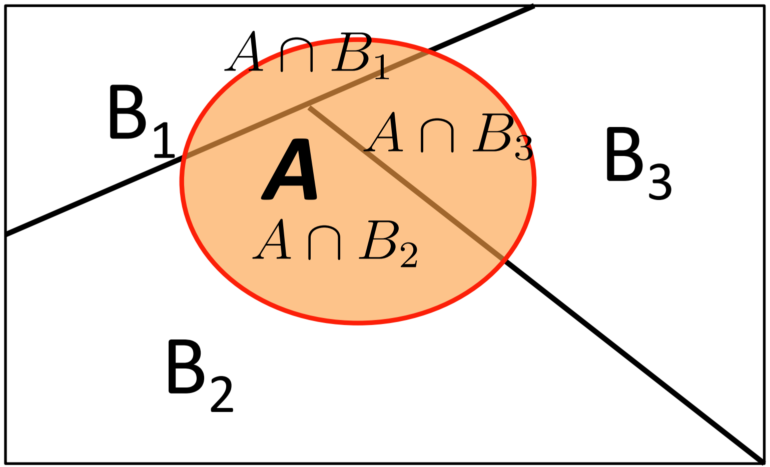
$$if\ B_i \cap B_j = \emptyset\ for\ all\ i \neq j$$



Bayes rule using total prob.

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{P(A)}$$

$$= \frac{P(A|B_j)P(B_j)}{\sum_j P(A|B_j)P(B_j)}$$



Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is 1/100,000. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is $P(D|T)$, the probability of having disease given a positive test result?

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

Using total prob.

$$= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

Bayes rule: rare disease test

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$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

Independence

- One definition:

$$P(A|B) = P(A) \ or$$
$$P(B|A) = P(B)$$

Whether A happened doesn’t change the probability of B and vice versa

Independence: example

- Suppose that we have a fair coin and it is tossed twice. let A be the event “the first toss is a head” and B the event “the two outcomes are the same.”

- These two events are independent!

Independence

- Alternative definition

$$P(A|B) = P(A)$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A \cap B) = P(A)P(B)}$$

Testing Independence:

- Suppose you draw one card from a standard deck of cards. E_1 is the event that the card is a King, Queen or Jack. E_2 is the event the card is a Heart. Are E_1 and E_2 independent?

Simulation of Conditional Probability

http://www.randomservices.org/random/apps/ConditionalProbabilityExperiment.html

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability"
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics"

Assignments

- Work on Module Week 3 on Canvas
- Next time: More on independence and conditional probability

See you next time

See You!