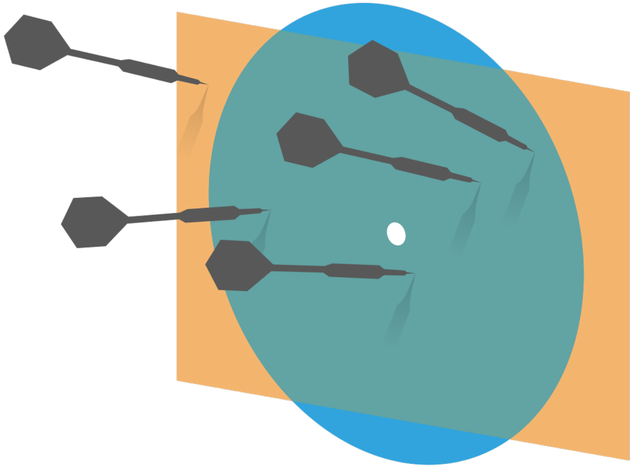
Probability and Statistics for Computer Science



Credit: wikipedia

Conditional probability comes back in matrix!

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Last time

- Markov Chain (I)

Objective

- Markov Chain (II)
- Q/A
- Concept review

Markov chain

- Markov chain is a process in which outcome of any trial in a sequence is **conditioned by the outcome of the trial immediately preceding, but not by earlier ones**.
- Such dependence is called **chain dependence**

Andrey Markov (1856-1922)

Markov chain in terms of probability

- Let $X_0, X_1, \ldots$ be a sequence of discrete finite-valued random variables

- The sequence is a Markov chain if the probability distribution X_t only depends on the distribution of the immediately preceding random variable X_{t-1}

$$P(X_t|X_0 \ldots, X_{t-1}) = P(X_t|X_{t-1})$$

- If the conditional probabilities (transition probabilities) do **NOT change with time**, it's called **constant Markov chain**.

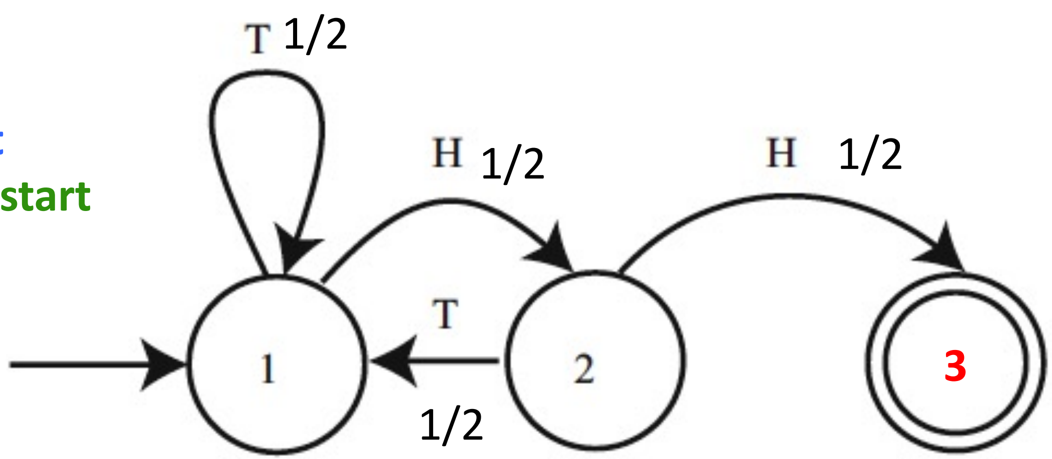
$$P(X_t|X_{t-1}) = P(X_{t-1}|X_{t-2}) = \ldots = P(X_1|X_0)$$

Coin example

- Toss a fair coin until you see two heads in a row and then stop, what is the probability of stopping after exactly **n** flips?

- Use a state diagram, which is a **directed graph**. Circles are the states of likely outcomes. Arrow directions show the direction of transitions. Numbers over the arrows show transition probabilities.

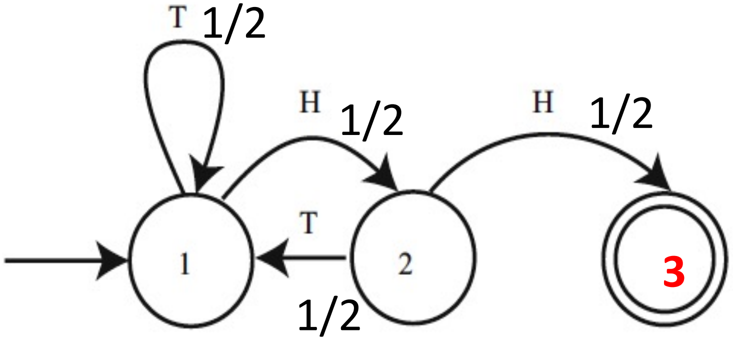
1 -> Start or just had tail/restart
2 -> had one head after start/restart
3 -> 2heads in a row/Stop



The model helps form recurrence formula

- Let p_n be the probability of stopping after **n** flips

$$p_1 = 0 \quad p_2 = 1/4 \quad p_3 = 1/8 \quad p_4 = 1/8 \quad \ldots$$



The model helps form recurrence formula

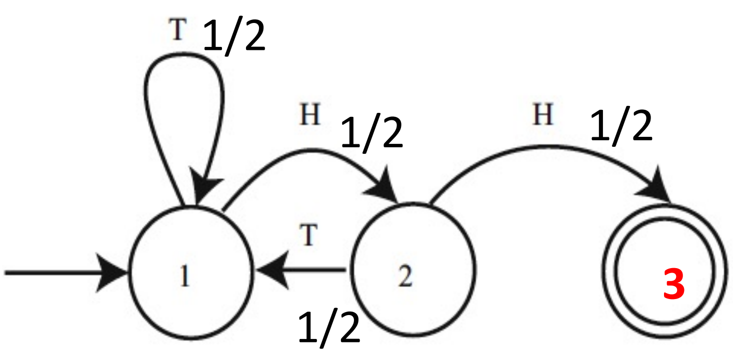
- Let p_n be the probability of stopping after **n** flips

$$p_1 = 0 \quad p_2 = 1/4 \quad p_3 = 1/8 \quad p_4 = 1/8 \quad \ldots$$

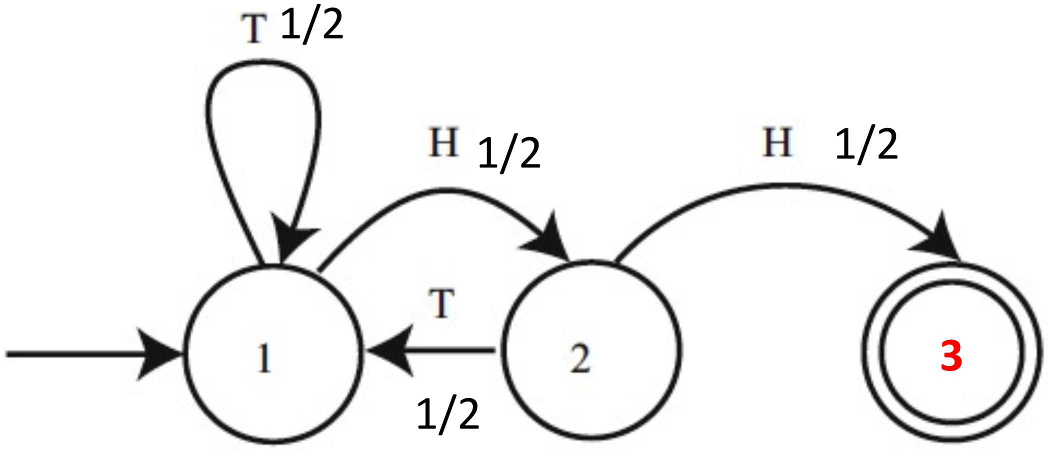
- If $n > 2$, there are two ways the sequence starts
  - Toss T and finish in n-1 tosses
  - Or toss HT and finish in n-2 tosses
- So, we can derive a recurrence relation

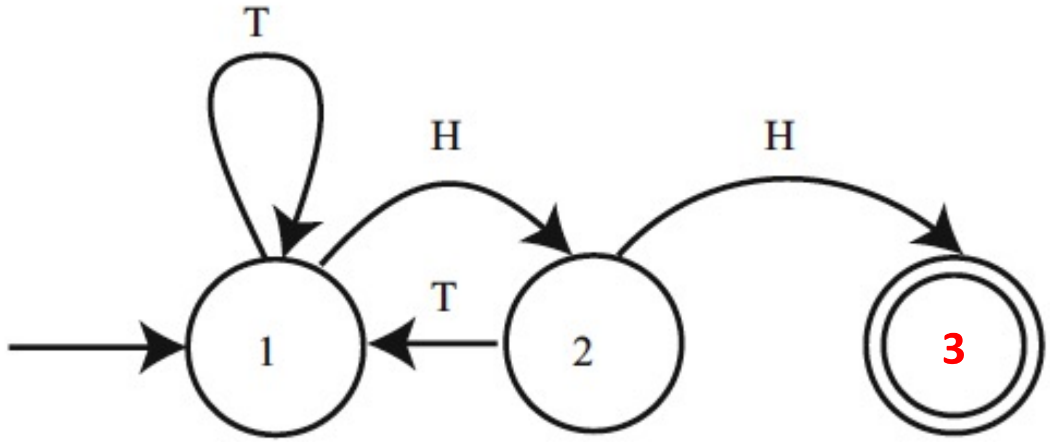
$$p_n = \frac{1}{2} p_{n-1} + \frac{1}{4} p_{n-2}$$

P(T) P(HT)



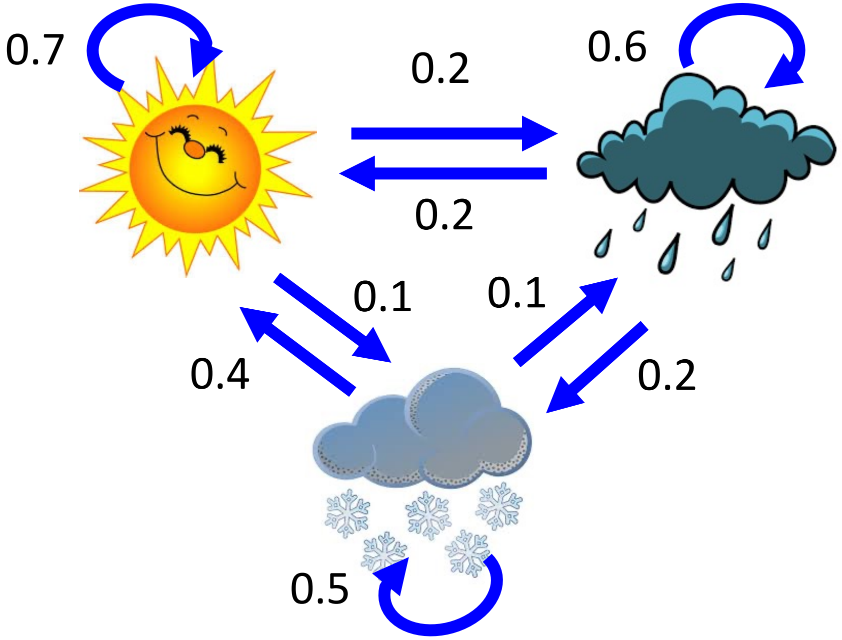
Transition probability btw states





Transition probability matrix: weather model

- Let's model daily weather as one of the three states (Sunny, Rainy, and Snowy) with Markov chain that has the transition probabilities as shown here.



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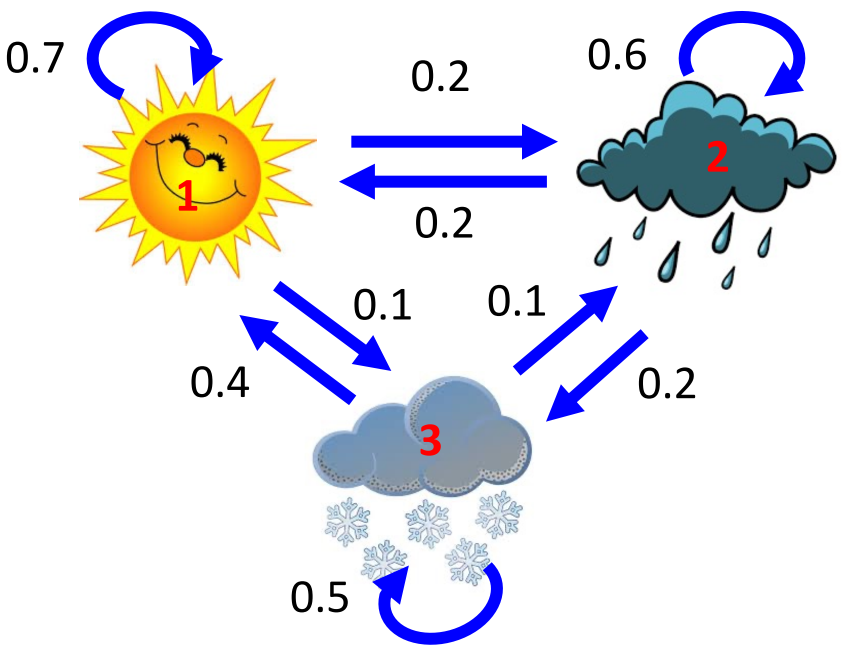


i, the current state at time point t
j, the next state at time point t+1

	Sunny	Rainy	Snowy	
	0.7	0.2	0.1	Sunny
$\boldsymbol{P} =$	0.2	0.6	0.2	Rainy
	0.4	0.1	0.5	Snowy

$$\boldsymbol{P} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

The transition probability matrix

Q: Is this TRUE?

For a constant Markov Chain, at any step t, the probability distribution among the states remain the same.

A. Yes.

B. No.

Q: The transition probabilities for a node sum to 1

A. Yes.

B. No.

Only the row sum is 1, that is: the probabilities associated with outgoing arrows sum to 1.

Transition probability matrix properties

- The transition probability matrix $\boldsymbol{P}$ is a square matrix with entries p_{ij}
- Since $p_{ij} = P(X_t = j | X_{t-1} = i)$

$$p_{ij} \geq 0 \quad \text{and} \quad \sum_j p_{ij} = 1$$

	Sunny	Rainy	Snowy
Sunny	0.7	0.2	0.1
Rainy	0.2	0.6	0.2
Snowy	0.4	0.1	0.5

$$\boldsymbol{P} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

The transition probability matrix

Probability distributions over states

- Let $\boldsymbol{\pi}$ be a row vector containing the probability distribution over all the finite discrete states at t=0

$$\pi_i = P(X_0 = i)$$

- For example: if it is rainy today, and today is t=0, then

$$\boldsymbol{\pi} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

- Let $\mathbf{P}^{(t)}$ be a row vector containing the probability distribution over states at time point t

$$\mathrm{p}_i^{(t)} = P(X_t = i)$$

Propagating the probability distribution

- Propagating from t=0 to t=1,

$$
\begin{aligned}
P_j^{(1)} &= P(X_1 = j) \\
&= \sum_i P(X_1 = j, X_0 = i) \\
&= \sum_i P(X_1 = j | X_0 = i) P(X_0 = i) \\
&= \sum_i p_{ij} \pi_i
\end{aligned}
$$

- In matrix notation,

$$
\boldsymbol{p}^{(1)} = \boldsymbol{\pi} P
$$

Probability distributions:

- Suppose that it is rainy, we have the initial probability distribution.

$$\boldsymbol{\pi} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

- What are the probability distributions for tomorrow and the day after tomorrow?

$$\boldsymbol{p}^{(1)} = \boldsymbol{\pi} P$$

$$\boldsymbol{p}^{(2)} = \boldsymbol{p}^{(1)} P$$

Propagating to t= ∞

- We have just seen that

$$\boldsymbol{p}^{(2)} = \boldsymbol{p}^{(1)} P = (\boldsymbol{\pi} P) P = \boldsymbol{\pi} P^2$$

- So, in general $\boldsymbol{p}^{(t)} = \boldsymbol{\pi} P^t$

- If one state can be reached from any other state in the graph, the Markov chain is called **irreducible** **(single chain)**.

- Furthermore , if it satisfies: $\lim_{t \to \infty} \boldsymbol{\pi} P^t = \boldsymbol{S}$

  then the Markov chain is stationary and **S** is the stationary distribution.

Stationary distribution

- The stationary distribution $\boldsymbol{S}$ has the following property: $\boldsymbol{s}P = \boldsymbol{s}$

- $\boldsymbol{S}$ is a row eigenvector of **P** with eigenvalue 1

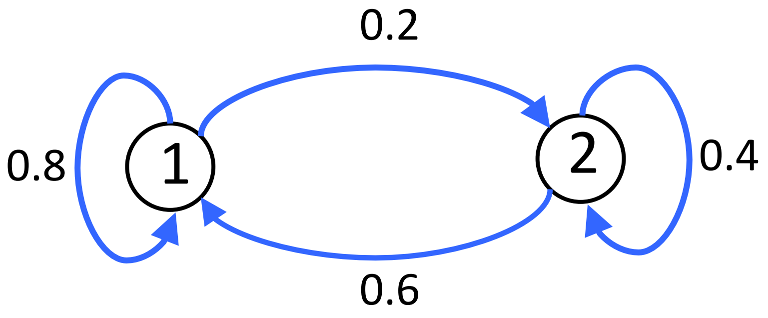
- In the example of the weather model, regardless of the initial distribution,

$$\boldsymbol{S} = \lim_{t \to \infty} \boldsymbol{\pi} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}^{\mathbf{t}} = \begin{bmatrix} \frac{18}{37} & \frac{11}{37} & \frac{8}{37} \end{bmatrix}$$

Example: Up-to-date or behind model

State 1: Up-to-date

State 2: Behind



What's the transition matrix?

If I start with π = [0, 1], what is my probability of being up-to-date eventually? 3/4

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

Example: Up-to-date or behind model

$$SP = S \Rightarrow (SP)^T = S^T \Rightarrow P^T S^T = S^T$$
$$(P^T - I)S^T = 0$$

$$\left(\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} - I \right) S^T = 0$$

let $S^T = u$

$$\begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix} u = 0 \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

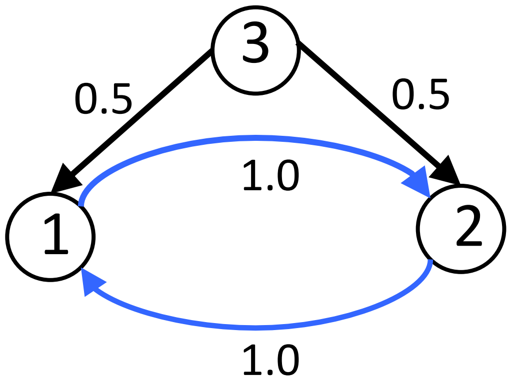
$$u_1 + u_2 = 1$$

$$\Rightarrow u_1 = \frac{3}{4} \qquad u_2 = \frac{1}{4}$$

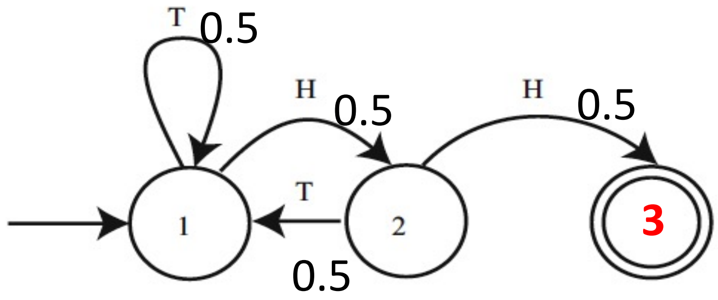
Example: Up-to-date or behind model

$$SP = S \Rightarrow (SP)^T = S^T \Rightarrow P^T S^T = S^T$$
$$(P^T - I)S^T = 0$$

Examples of non-stationary Markov chains



Periodic

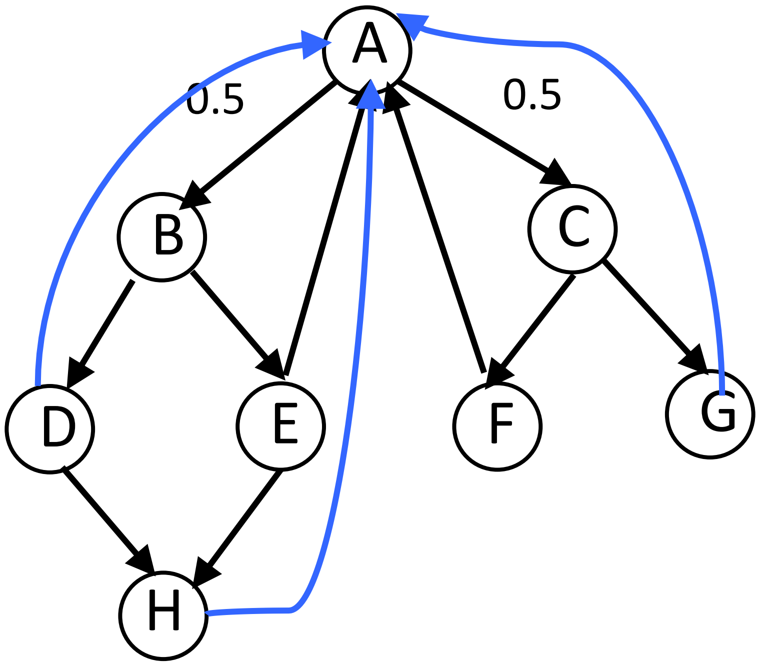


Absorbing

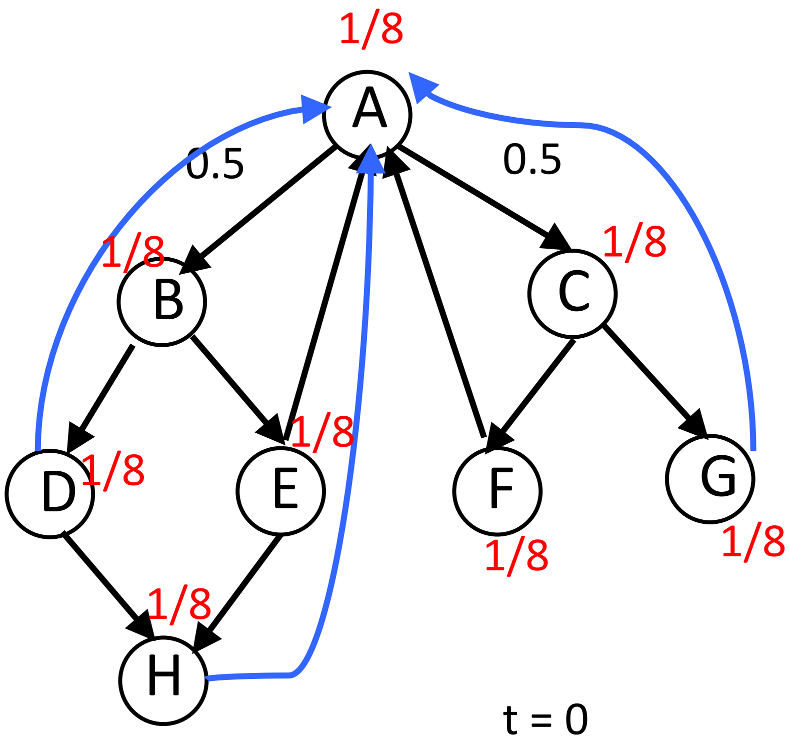
PageRank Example

- How to rate web pages objectively?
- The PageRank algorithm by Page et al. made **Google** successful
- The method utilized **Markov chain model** and applied it to the large list of webpages.
- To illustrate the point, we use a small-size example and assume a simple **stationary model**.

Suppose we are randomly surfing a network of webpages



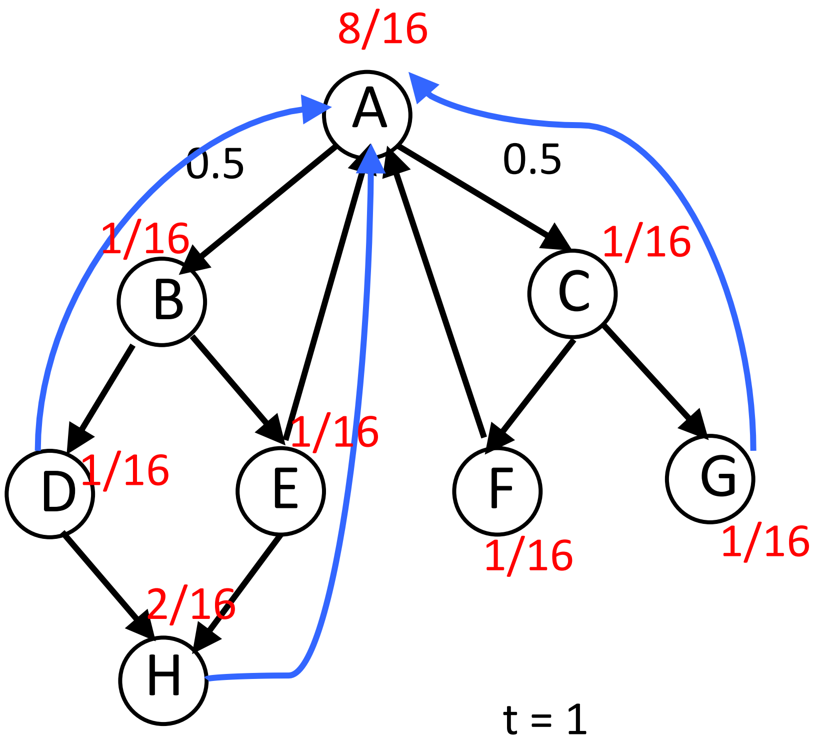
Initialize the distribution uniformly



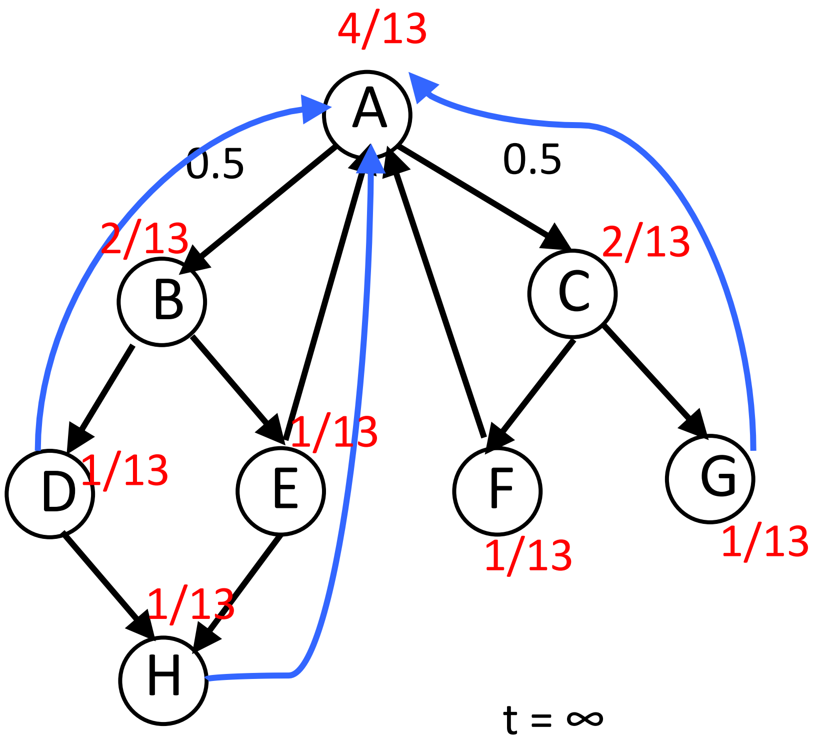
$$\boldsymbol{\pi} = \begin{bmatrix} 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Update the distribution iteratively



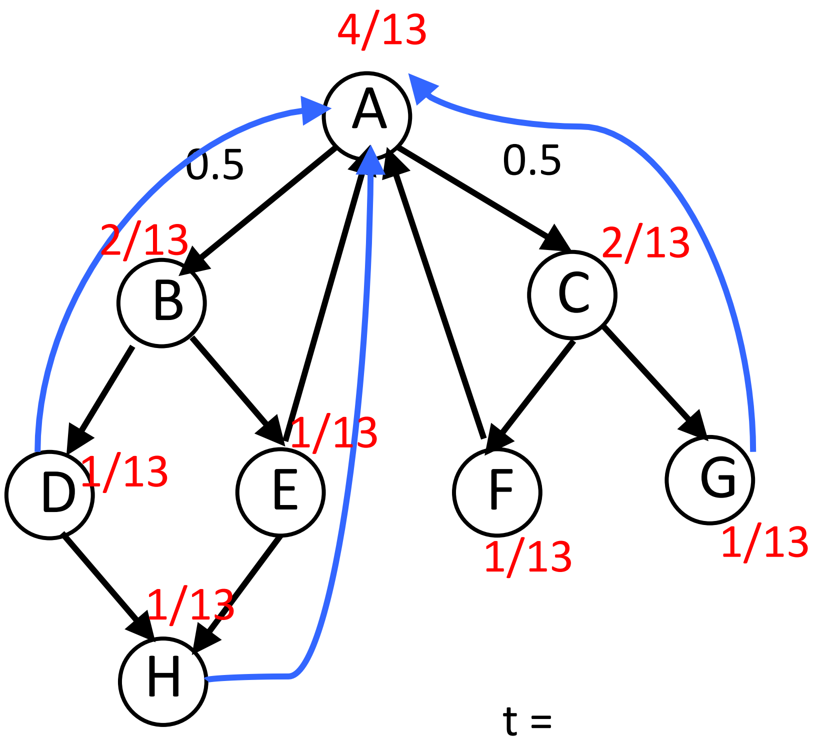
Until the stationary distribution



If the surfer get trapped

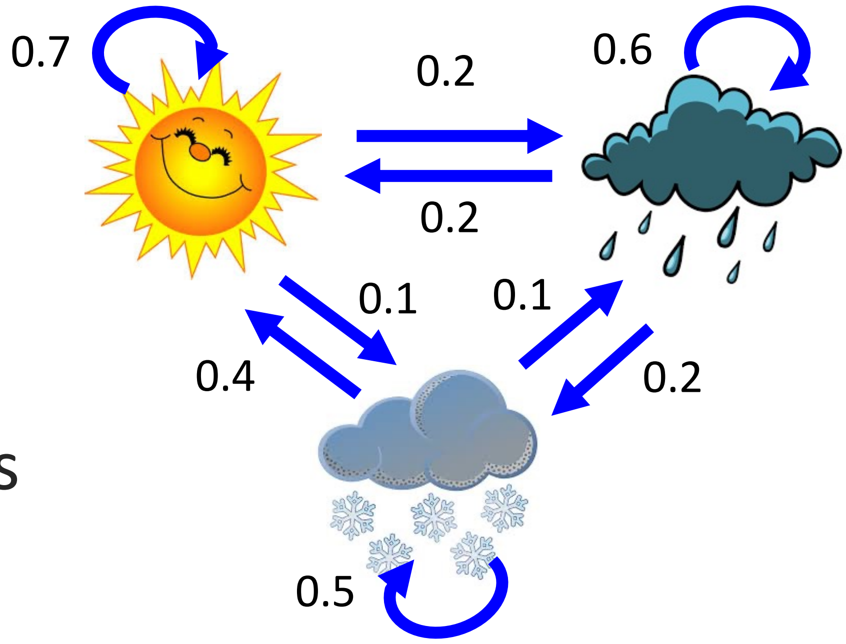
- Allow “teleport” with small probability from any page to another

- Or allow “teleport” with user input of URL



Diverse applications of Markov Model

- Communication network
- Queue modeling
- DNA sequence modeling
- Natural language processing
- Single-cell large data analysis
- Financial/Economic model
- Music



Final Exam

- Time: 1:30pm 12/13 Mon. Central Time
- Duration: 3hrs
- Content coverage: Ch1-14, except 8, details are on Canvas
- Open book and lecture notes
- Format: 50 multiple choices, on PrairieLearn proctored by Staff on Zoom

Additional References

- Robert V. Hogg, Elliot A. Tanis and Dale L. Zimmerman. “Probability and Statistical Inference”
- Kelvin Murphy, “Machine learning, A Probabilistic perspective”

Acknowledgement

Thank You!