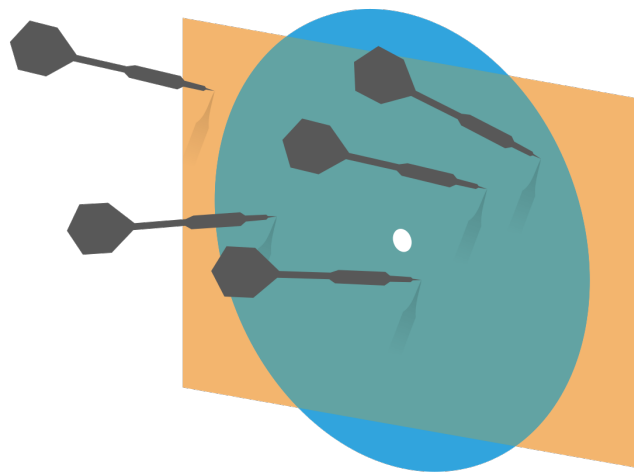


Probability and Statistics for Computer Science



Can we call e the
exciting e ?

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n$$

Credit: wikipedia

Last time

- ✱ **Poisson** distribution
- ✱ Continuous random variable; uniform distribution
- ✱ **Exponential** distribution

Objectives

✱ Exponential distribution

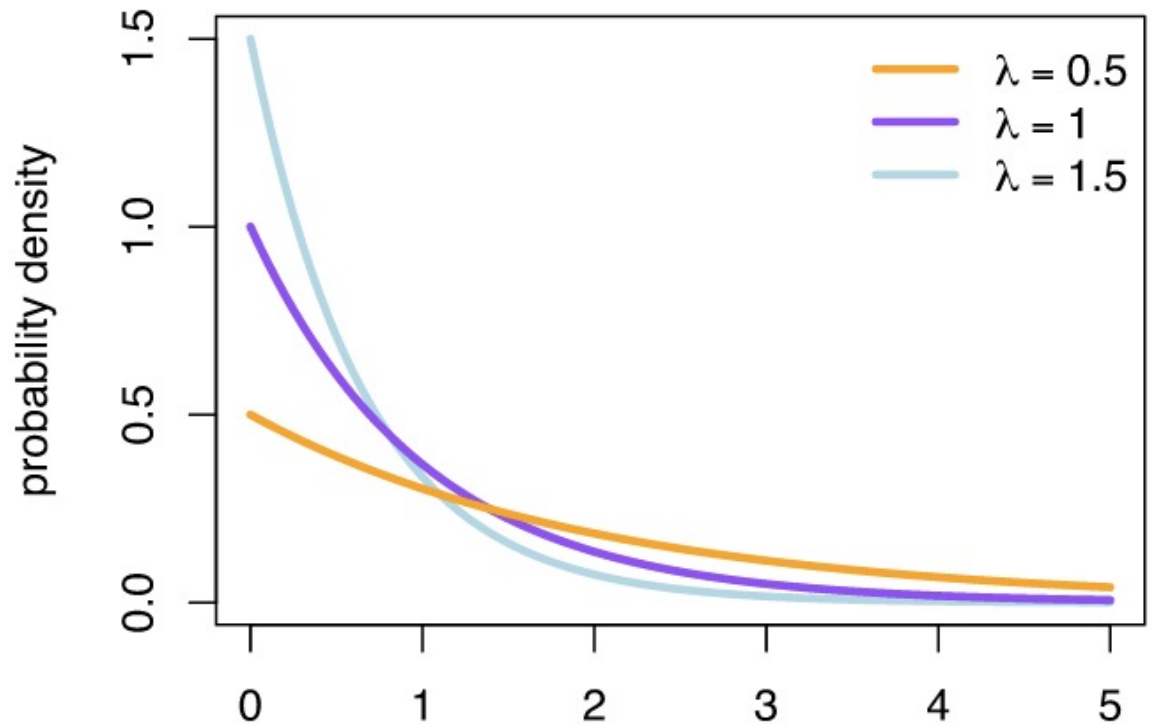
✱ Normal (Gaussian) distribution

Exponential distribution

- ✱ Common Model for waiting time

- ✱ Associated with the Poisson distribution with the same λ

$$p(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$



Exponential distribution

- ✱ A continuous random variable X is exponential if it represent the “time” until next incident in a Poisson distribution with intensity λ . Proof See Degroot et al Pg 324.

$$p(x) = \lambda e^{-\lambda x} \quad for \ x \geq 0$$

- ✱ It's **similar to Geometric distribution** – the discrete version of waiting in queue

Expectations of Exponential distribution

- ✱ A continuous random variable X is exponential if it represent the “time” until next incident in a Poisson distribution with intensity λ .

$$p(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$

$$E[X] = \frac{1}{\lambda} \quad \& \quad \text{var}[X] = \frac{1}{\lambda^2}$$

Example of exponential distribution

- ✱ How long will it take until the next call to be received by a call center? Suppose it's a random variable T . If the number of incoming call is a Poisson distribution with intensity $\lambda = 20$ in an hour. What is the expected time for T ?

Q:

✱ A store has a number of customers coming on Sat. that can be modeled as a Poisson distribution. In order to measure the average rate of customers in the day, the staff recorded the time between the arrival of customers, can he reach the same goal?

A. Yes B. No

Normal (Gaussian) distribution

- ✱ The most famous continuous random variable distribution. The probability density is this:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$



Carl F. Gauss
(1777-1855)
Credit: wikipedia

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$$E[X] = \mu \quad \& \quad var[X] = \sigma^2$$

Normal (Gaussian) distribution

- ✱ The most famous continuous random variable distribution.

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

$$\int_{-\infty}^{+\infty} p(x) dx = 1$$

$$E[X] = \mu \quad \& \quad \text{var}[X] = \sigma^2$$



Carl F. Gauss
(1777-1855)
Credit: wikipedia

Normal (Gaussian) distribution

- ✱ A lot of data in nature are approximately normally distributed, ie. **Adult height**, etc.

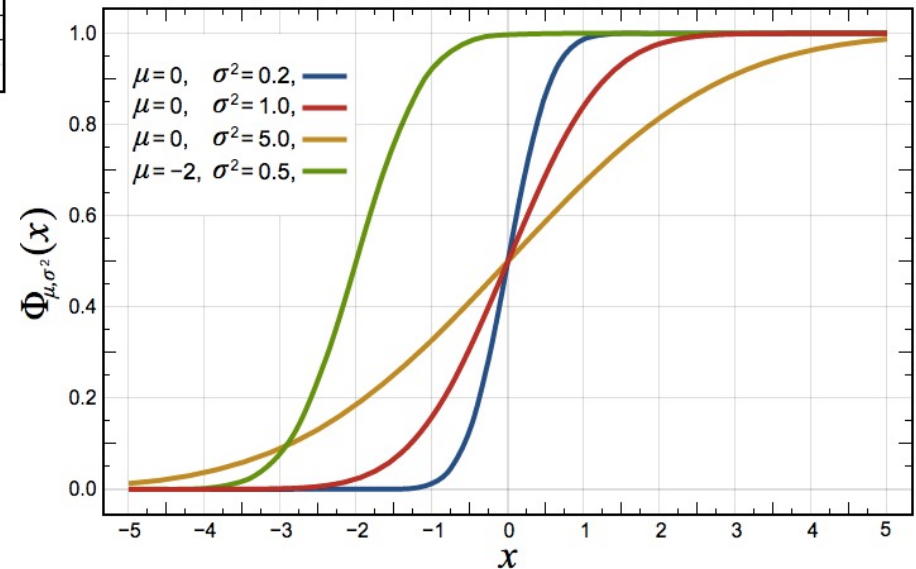
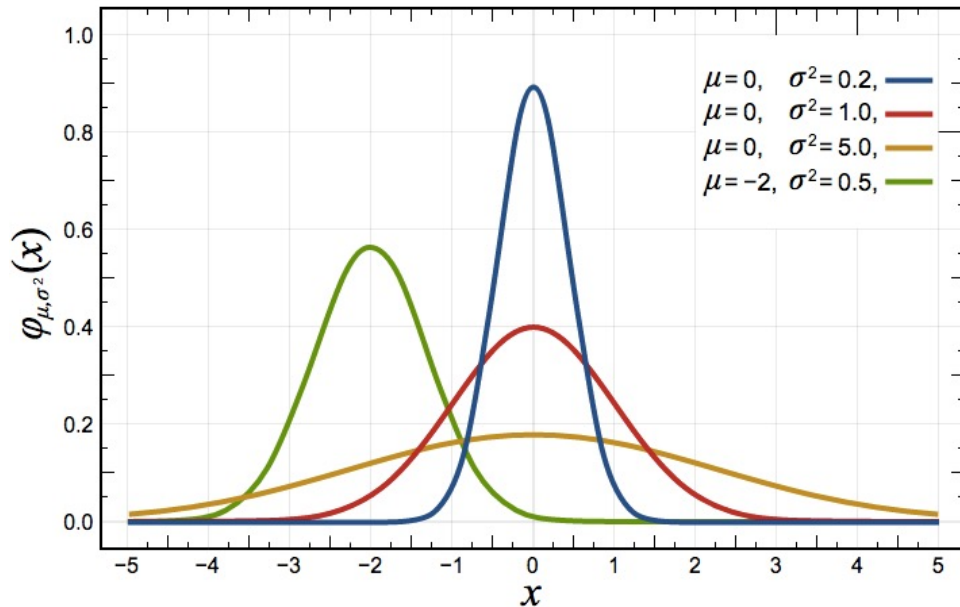
$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$



Carl F. Gauss
(1777-1855)
Credit: wikipedia

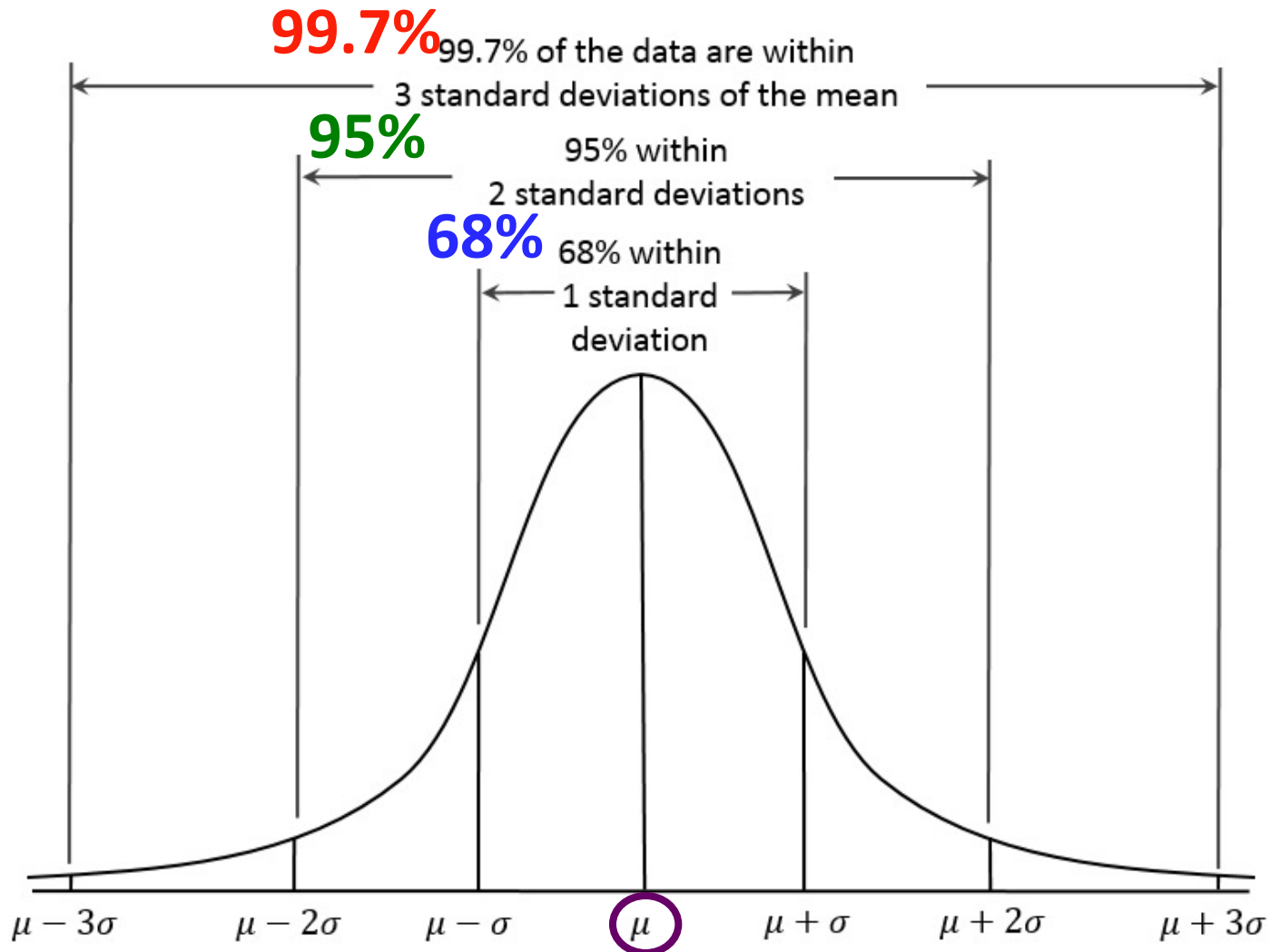
$$E[X] = \mu \quad \& \quad var[X] = \sigma^2$$

PDF and CDF of normal distribution curves



Credit: wikipedia

Spread of normal (Gaussian) distributed data



Credit:
wikipedia

Standard normal distribution

- ✱ If we standardize the normal distribution (by subtracting μ and dividing by σ), we get a random variable that has standard normal distribution.
- ✱ A continuous random variable X is **standard normal** if

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Derivation of standard normal distribution

$$\begin{aligned} & \int_{-\infty}^{+\infty} p(x) \, dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \exp\left(-\frac{\hat{x}^2}{2}\right) \cancel{\sigma} \, d\hat{x} \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\hat{x}^2}{2}\right) d\hat{x} \\ &= \int_{-\infty}^{+\infty} p(\hat{x}) \, d\hat{x} \end{aligned}$$

$\hat{x} = \frac{x - \mu}{\sigma}$

Call this standard and omit using a **hat**

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Q. What is the mean of standard normal?

A. 0

B. 1

Q. What is the standard deviation of standard normal?

A. 0

B. 1

Standard normal distribution

- ✱ If we standardize the normal distribution (by subtracting μ and dividing by σ), we get a random variable that has standard normal distribution.
- ✱ A continuous random variable X is **standard normal** if

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$E[X] = 0 \quad \& \quad \text{var}[X] = 1$$

Another way to check the spread of normal distributed data

- ✱ Fraction of **normal** data within **1** standard deviation from the mean.

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx \simeq 0.68$$

- ✱ Fraction of **normal** data within **k** standard deviations from the mean.

$$\frac{1}{\sqrt{2\pi}} \int_{-k}^k \exp\left(-\frac{x^2}{2}\right) dx$$

Using the standard normal's table to calculate for a normal distribution's probability

✱ If $X \sim N(\mu=3, \sigma^2=16)$ (normal distribution)

$$P(X \leq 5) = ?$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015

Q.

✱ If $X \sim N(\mu=3, \sigma^2=16)$ (normal distribution)

$P(X \leq 5) = ?$ A . 0.5199 B. 0.5987 C. 0.6915

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015

Q. Is the table with only positive x values enough?

A. Yes B. No.

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319

Central limit theorem (CLT)

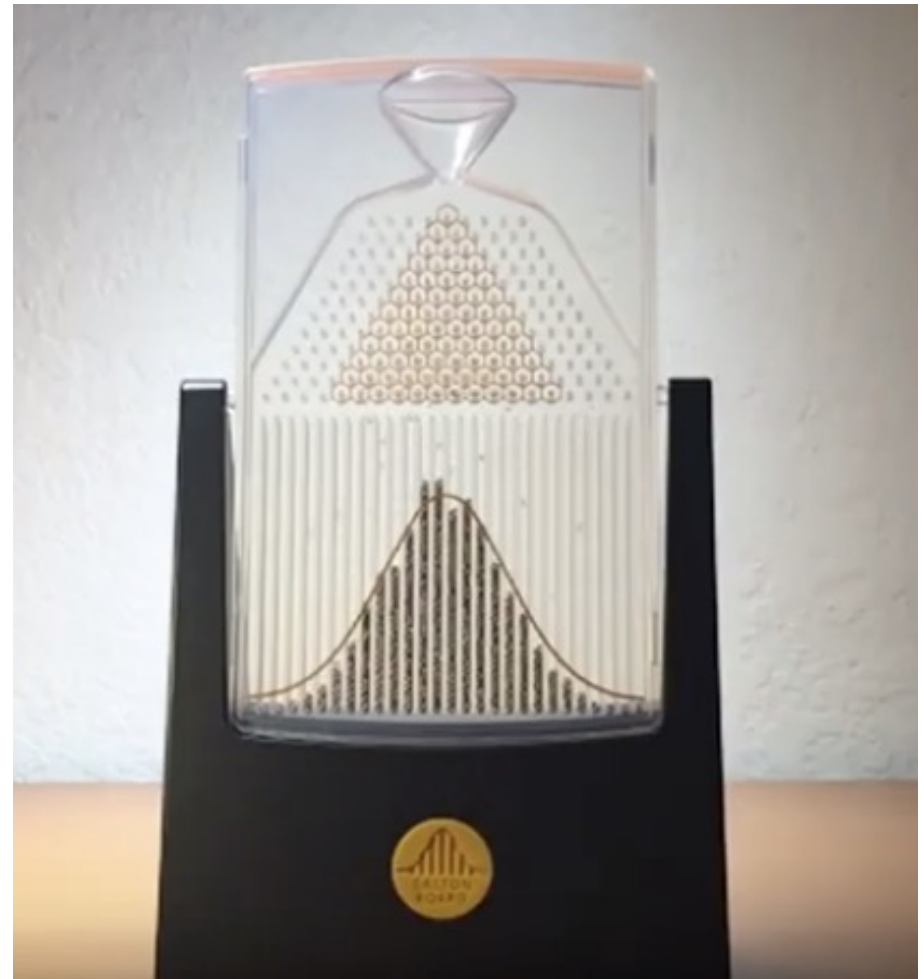
- ✱ The distribution of the **sum** of **N** independent identical (IID) random variables tends toward a **normal** distribution as **$N \longrightarrow \infty$**
- ✱ Even when the component random variables are not exactly IID, the result is approximately true and very useful in practice

Central limit theorem (CLT)

- ✱ CLT helps explain the prevalence of normal distributions in nature
- ✱ A binomial random variable tends toward a normal distribution when **N** is large due to the fact it is the sum of IID Bernoulli random variables

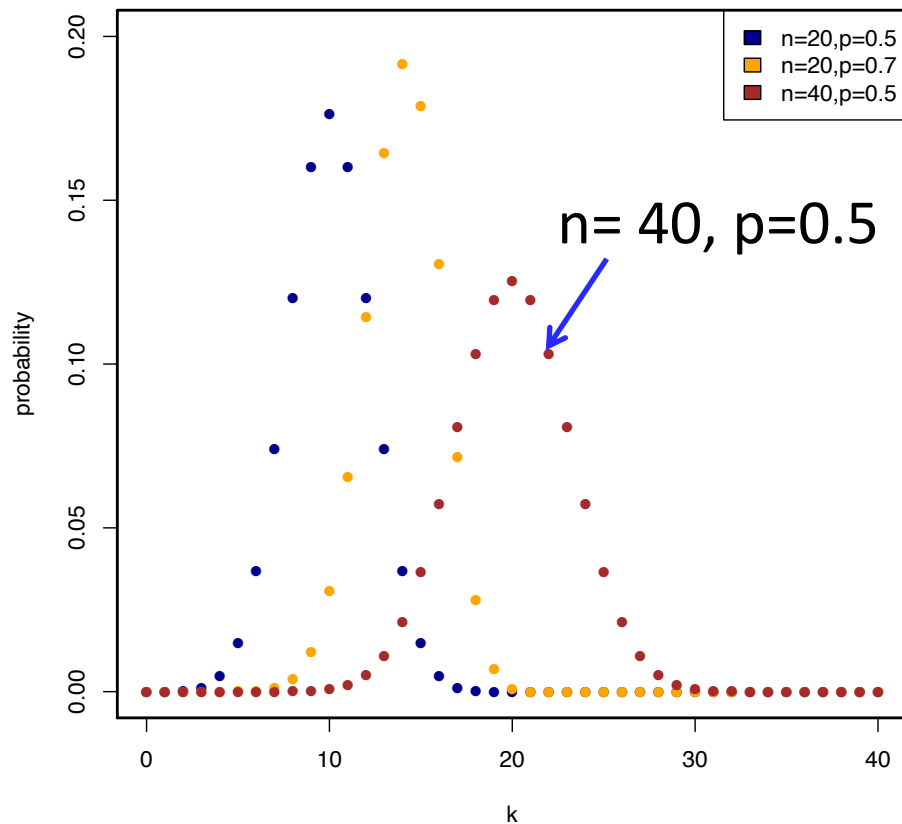
The Binomial distributed beads of the Galton Board

The **Binomial** distribution looks very similar to **Normal** when N is large

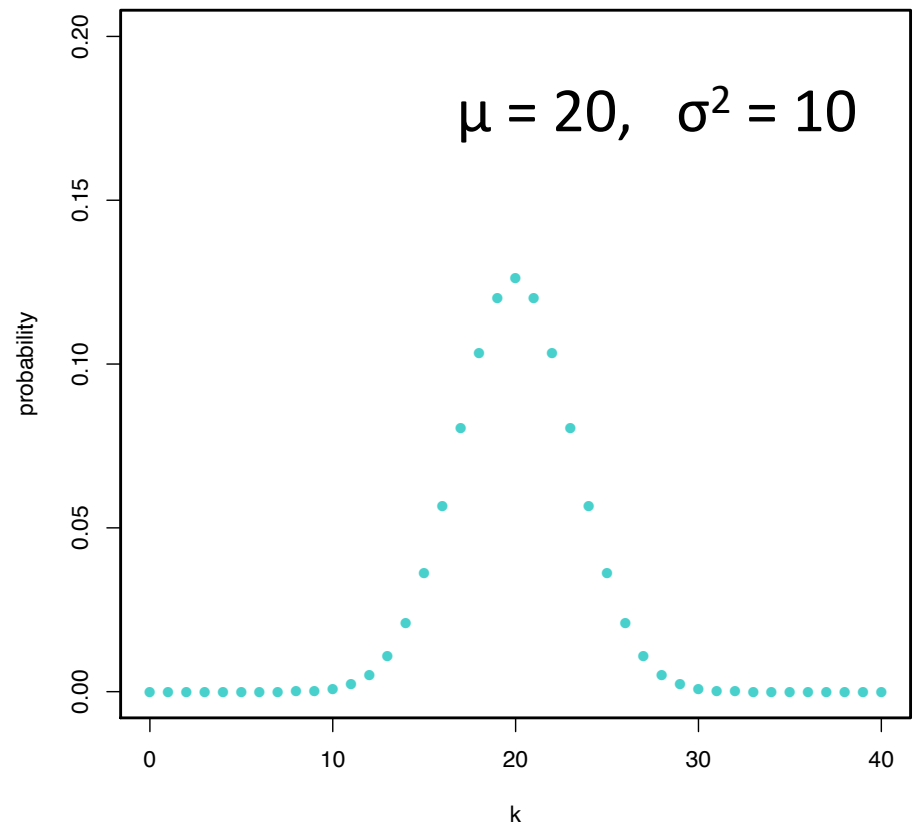


Binomial approximation with Normal

Binomial distribution



Approximation with Normal



Binomial approximation with Normal

- ✱ Let k be the number of heads appeared in 40 tosses of fair coin
- ✱ The goal is to estimate the following with normal

$$P(10 \leq k \leq 25) = \sum_{k=10}^{25} \binom{40}{k} 0.5^k 0.5^{40-k}$$

$$= \sum_{k=10}^{25} \binom{40}{k} 0.5^{40} \simeq 0.96$$

$$E[k] = np = 40 \cdot 0.5 = 20 \qquad \begin{aligned} std[k] &= \sqrt{np(1-p)} \\ &= \sqrt{40 \cdot 0.5 \cdot 0.5} = \sqrt{10} \end{aligned}$$

Binomial approximation with Normal

- ✱ Use the same mean and standard deviation of the original binomial distribution.

$$\mu = 20 \qquad \sigma = \sqrt{10} \simeq 3.16$$

- ✱ Then standardize the normal to do the calculation

$$\begin{aligned} P(10 \leq k \leq 25) &\simeq \frac{1}{\sigma\sqrt{2\pi}} \int_{10}^{25} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{10-20}{3.16}}^{\frac{25-20}{3.16}} \exp\left(-\frac{x^2}{2}\right) dx \\ &\simeq 0.94 \end{aligned}$$

Assignments this week

- ✱ Week 6 quiz
- ✱ HW5
- ✱ Prepare for Midterm1:
 - ✱ Practice exams
 - ✱ Read through instructions on Canvas

Additional References

- ✱ Charles M. Grinstead and J. Laurie Snell
"Introduction to Probability"
- ✱ Morris H. Degroot and Mark J. Schervish
"Probability and Statistics"

See you next time

*See
You!*

