Truncation errors: using Taylor series to approximation functions
Approximating functions using polynomials:

Let’s say we want to approximate a function $f(x)$ with a polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

For simplicity, assume we know the function value and its derivatives at $x_0 = 0$ (we will later generalize this for any point). Hence,

$$f'(x) = a_1 + 2 a_2 x + 3 a_3 x^2 + 4 a_4 x^3 + \cdots$$

$$f''(x) = 2 a_2 + (3 \times 2) a_3 x + (4 \times 3) a_4 x^2 + \cdots$$

$$f'''(x) = (3 \times 2) a_3 + (4 \times 3 \times 2) a_4 x + \cdots$$

$$f''''(x) = (4 \times 3 \times 2) a_4 + \cdots$$

$$f(0) = a_0 \quad f''(0) = 2 a_2 \quad f'''(0) = (4 \times 3 \times 2) a_4$$

$$f'(0) = a_1 \quad f''''(0) = (3 \times 2) a_3$$

$$f^{(i)}(0) = i! \ a_i$$
Taylor Series

Taylor Series approximation about point $x_0 = 0$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!}x^i$$
Taylor Series

In a more general form, the Taylor Series approximation about point $x_o$ is given by:

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f'''(x_o)}{3!}(x - x_o)^3 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_o)}{i!} (x - x_o)^i$$
Iclicker question

Assume a finite Taylor series approximation that converges everywhere for a given function $f(x)$ and you are given the following information:

$$f(1) = 2; f'(1) = -3; f''(1) = 4; f^{(n)}(1) = 0 \forall n \geq 3$$

Evaluate $f(4)$

A) 29
B) 11
C) -25
D) -7
E) None of the above
Taylor Series

We cannot sum infinite number of terms, and therefore we have to truncate.

How big is the error caused by truncation? Let’s write $h = x - x_o$

$$f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!} (h)^i = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_o)}{i!} (h)^i$$

And as $h \to 0$ we write:

$$\left| f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!} (h)^i \right| \leq C \cdot h^{n+1}$$

Error due to Taylor approximation of degree n

$$\left| f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!} (h)^i \right| = O(h^{n+1})$$
Taylor series with remainder

Let \( f \) be \((n + 1)\)-times differentiable on the interval \((x_o, x)\) with \( f^{(n)} \) continuous on \([x_o, x]\), and \( h = x - x_o \)

\[
f(x_o + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!} (h)^i = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_o)}{i!} (h)^i
\]

Then there exists a \( \xi \in (x_o, x) \) so that

\[
f(x_o + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!} (h)^i = \frac{f^{(n+1)}(\xi)}{(n + 1)!} (\xi - x_o)^{n+1}
\]

\( f(x) - T(x) = R(x) \)

And since \(|\xi - x_o| \leq h\)

\[
f(x_o + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!} (h)^i \leq \frac{f^{(n+1)}(\xi)}{(n + 1)!} (h)^{n+1}
\]
Demo: Polynomial Approximation with Derivatives

\[ f = \sqrt{-x^2 + 1} \]

\[ \text{taylor} = -\frac{x^2}{2} + 1 \]

\[ \text{error} = \text{taylor} - f \]
Demo: Polynomial Approximation with Derivatives

\[
f = \sqrt{-x^2 + 1} \quad \text{taylor} \quad -\frac{x^2}{2} + 1
\]

\[
\text{error} = \text{taylor} - f
\]
Iclicker question

Error Order for Taylor series

The series expansion for \( e^x \) about 2 is

\[
\exp(2) \cdot \left( 1 + (x - 2) + \frac{(x - 2)^2}{2!} + \frac{(x - 2)^3}{3!} + \ldots \right).
\]

If we evaluate \( e^x \) using only the first four terms of this expansion (i.e. only terms up to and including \( \frac{(x-2)^3}{3!} \)), then what is the error in big-O notation?

**Choice**

A) \( O(x^4) \)
B) \( O(x^5) \)
C) \( O(x^3) \)
D) \( O((x - 2)^3) \)
E) \( O((x - 2)^4) \)

Demo “Taylor of exp(x) about 2”
Making error predictions

Suppose you expand $\sqrt{x} - 10$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.5$, you find that the Taylor truncation error is about $10^{-4}$.

What is the Taylor truncation error for $h_2 = 0.25$?

$\text{Error}(h) = O(h^{n+1})$, where $n = 3$, i.e.

$\text{Error}(h_1) \approx C \cdot h_1^4$

$\text{Error}(h_2) \approx C \cdot h_2^4$

While not knowing $C$ or lower order terms, we can use the ratio of $h_2/h_1$

$\text{Error}(h_2) \approx C \cdot h_2^4 = C \cdot h_1^4 \left(\frac{h_2}{h_1}\right)^4 \approx \text{Error}(h_1) \cdot \left(\frac{h_2}{h_1}\right)^4$

Can make prediction of the error for one $h$ if we know another.
Using Taylor approximations to obtain derivatives

Let’s say a function has the following Taylor series expansion about \( x = 2 \).

\[
f(x) = \frac{5}{2} - \frac{5}{2} (x - 2)^2 + \frac{15}{8} (x - 2)^4 - \frac{5}{4} (x - 2)^6 + \frac{25}{32} (x - 2)^8 + O((x - 2)^9)
\]

Therefore the Taylor polynomial of order 4 is given by

\[
t(x) = \frac{5}{2} - \frac{5}{2} (x - 2)^2 + \frac{15}{8} (x - 2)^4
\]

where the first derivative is

\[
t'(x) = -5(x - 2) + \frac{15}{2} (x - 2)^3
\]
Using Taylor approximations to obtain derivatives

We can get the approximation for the derivative of the function $f(x)$ using the derivative of the Taylor approximation:

$$t'(x) = -5(x - 2) + \frac{15}{2} (x - 2)^3$$

For example, the approximation for $f'(2.3)$ is

$$f'(2.3) \approx t'(2.3) = -1.2975$$

(note that the exact value is

$$f'(2.3) = -1.31444$$

What happens if we want to use the same method to approximate $f'(3)$?
The function

\[ f(x) = \cos(x) x^2 + \frac{\sin(2x)}{(x+2x^2)^3} \]

is approximated by the following Taylor polynomial of degree \( n = 2 \) about \( x = 2\pi \)

\[ t_2(x) = 39.4784 + 12.5664 (x - 2\pi) - 18.73922 (x - 2\pi)^2 \]

Determine an approximation for the first derivative of \( f(x) \) at \( x = 6.1 \)

A) 18.7741
B) 12.6856
C) 19.4319
D) 15.6840
Computing integrals using Taylor Series

A function $f(x)$ is approximated by a Taylor polynomial of order $n$ around $x = 0$.

$$t_n = \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i$$

We can find an approximation for the integral $\int_s^t f(x)\,dx$ by integrating the polynomial:

$$\int_s^t f(x)\,dx \approx \int_s^t a_0 + a_1 x + a_2 x^2 + a_3 x^3\,dx$$

$$= a_0 \int_s^t 1\,dx + a_1 \int_s^t x\,dx + a_2 \int_s^t x^2\,dx + a_3 \int_s^t x^3\,dx$$

Where we can use $\int_s^t x^i\,dx = \frac{t^{i+1}}{i+1} - \frac{s^{i+1}}{i+1}$

Demo “Computing PI with Taylor”
A function $f(x)$ is approximated by the following Taylor polynomial:

$$t_5(x) = 10 + x - 5x^2 - \frac{x^3}{2} + \frac{5x^4}{12} + \frac{x^5}{24} - \frac{x^6}{72}$$

Determine an approximated value for $\int_{-3}^{1} f(x) \, dx$

A) -10.27 
B) -11.77 
C) 11.77 
D) 10.27