Solving Linear System of Equations

The "Undo" button for Linear Operations

Matrix-vector multiplication: given the data $\boldsymbol{x}$ and the operator $\boldsymbol{A}$, we can find $\boldsymbol{y}$ such that

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

$$\boldsymbol{x} \xrightarrow[\text{transformation}]{\boldsymbol{A}} \boldsymbol{y}$$

What if we know $\boldsymbol{y}$ but not $\boldsymbol{x}$? How can we "undo" the transformation?

$$x = A^{-1} y$$

$$\boldsymbol{y} \xrightarrow[?]{\boldsymbol{A}^{-1}} \boldsymbol{x}$$

Solve $\boxed{\boldsymbol{A}}\boldsymbol{x} = \boxed{\boldsymbol{y}}$ **for** $\boldsymbol{x}$

Image Blurring Example

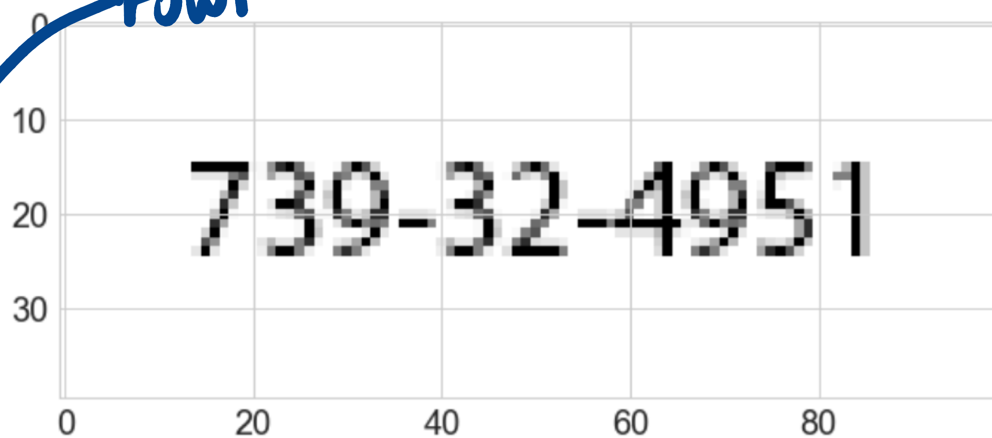
Demo "Image Blur Inverse"

row 0

row1

$\begin{bmatrix} A \end{bmatrix}$

m×n

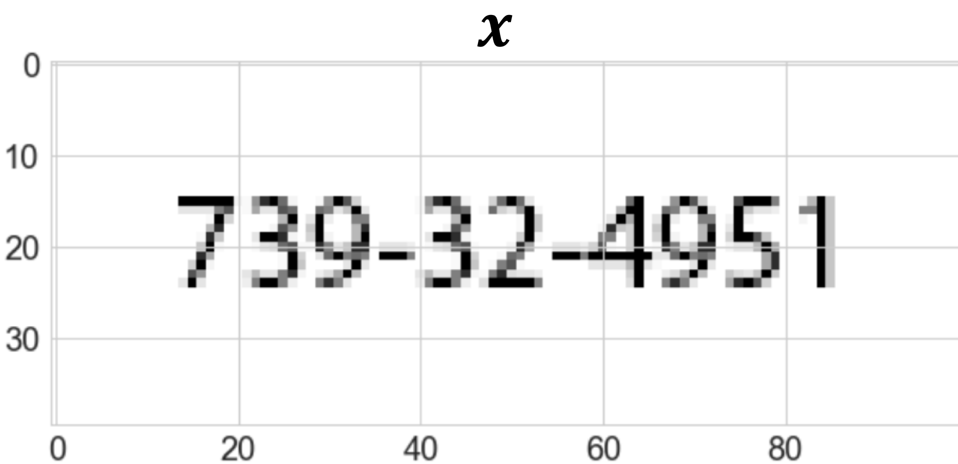


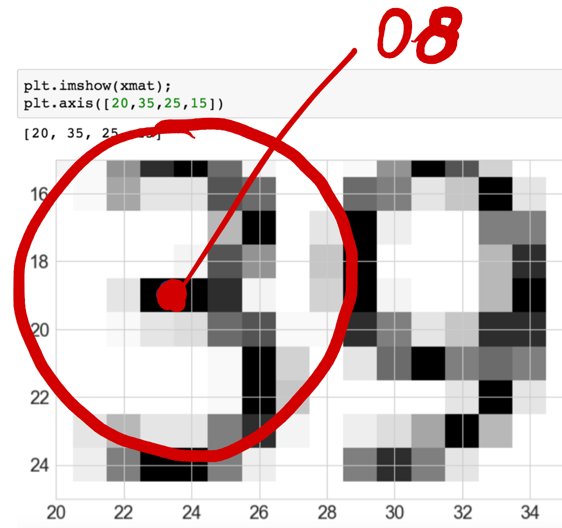
- Image is stored as a 2D array of real numbers between 0 and 1
  (0 represents a white pixel, 1 represents a black pixel)
- $\boldsymbol{xmat}$ contains the 2D data (the image) with dimensions 100x40
- Flatten the 2D array as a 1D array
- $\boldsymbol{x}$ contains the 1D data with dimension 4000,
- Apply blurring operation to data $\boldsymbol{x}$, i.e

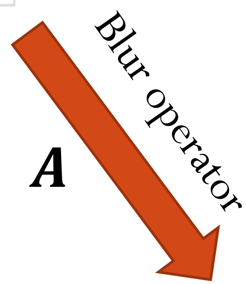
$$\boxed{\boldsymbol{y}} = \boxed{\boldsymbol{A}}\,\boldsymbol{x}$$

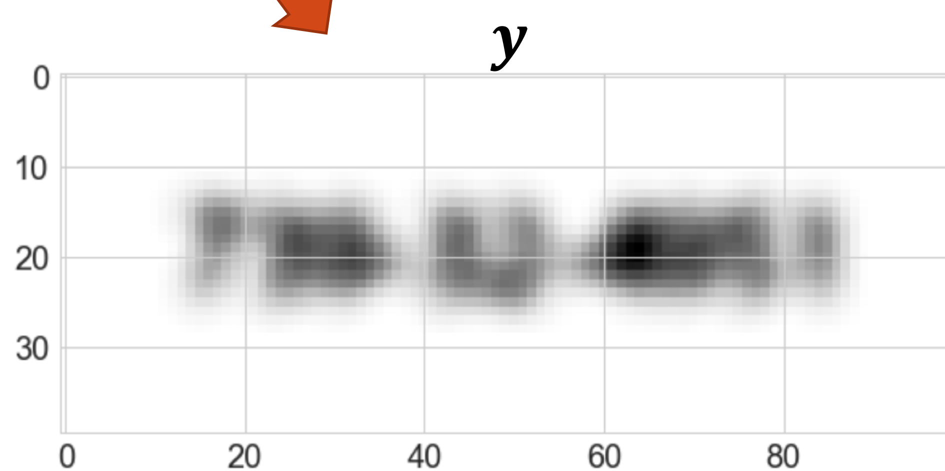
where $\boldsymbol{A}$ is the blur operator and $\boldsymbol{y}$ is the blurred image

Blur operator









blurred image (4000,)

Blur operator (4000,4000)

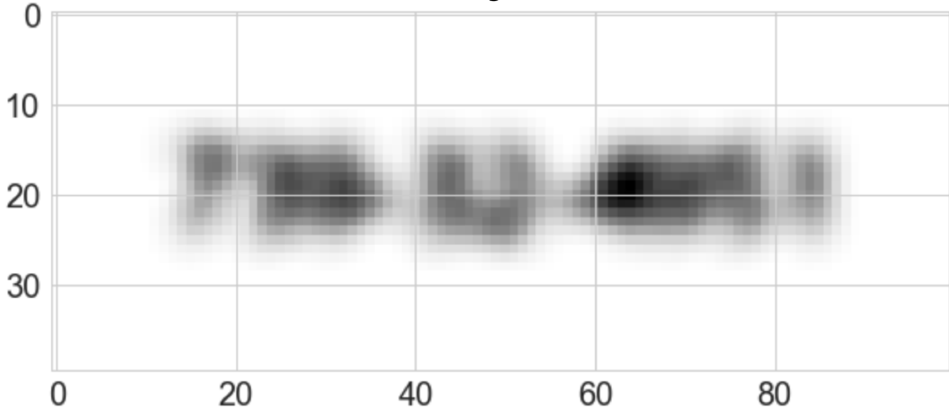
"original" image (4000,)

$$y = A\,x$$

"Undo" Blur to recover original image

DEMO

$\boldsymbol{y}$

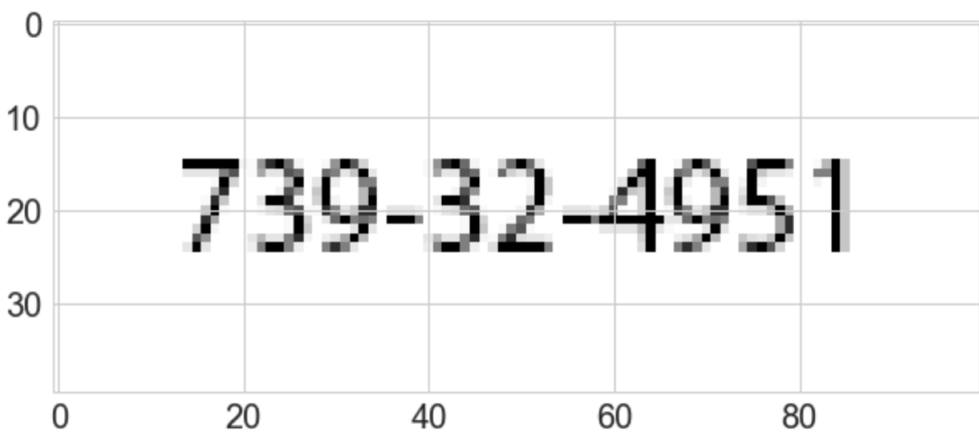


Solve

$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{y}$

for $\boldsymbol{x}$

$\boldsymbol{x}$



Assumptions:

1. we know the blur operator $\boldsymbol{A}$
2. the data set $\boldsymbol{y}$ does not have any noise ("clean data"

What happens if we add some noise to $\boldsymbol{y}$?

Linear System of Equations

How do we actually solve $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$?

We can start with an "easier" system of equations…

Let's consider triangular matrices (lower and upper):

$$\begin{pmatrix} L_{11} & 0 & \dots & 0 \\ L_{21} & L_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} U_{11} & U_{12} & \dots & U_{1n} \\ 0 & U_{22} & \dots & U_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & U_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Example: Forward-substitution for lower triangular systems

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 2 & 6 & 0 \\ 1 & 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 6 \\ 4 \end{pmatrix}$$

$2x_1 = 2 \longrightarrow x_1 = 1$

$3x_1 + 2x_2 = 2 \longrightarrow x_2 =$

$1x_1 + 2x_2 + 6x_3 = 6 \longrightarrow x_3 =$

$1x_1 + 3x_2 + 4x_3 + 2x_4 = 4 \longrightarrow x_4 =$

Example:

$$\begin{pmatrix} 2 & 8 & 4 & 2 \\ 0 & 4 & 4 & 3 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \\ 1 \end{pmatrix}$$

$$x_4 = \frac{1}{2}$$

$$x_3 = \frac{4 - 2\,\frac{1}{2}}{6} = \frac{1}{2}$$

$$x_2 = \frac{4 - 4\,\frac{1}{2} - 3\,\frac{1}{2}}{4} = \frac{1/2}{4} = \frac{1}{8}$$

$$x_1 = \frac{2 - 8\,\frac{1}{8} - 4\,\frac{1}{2} - 2\,\frac{1}{2}}{2} = \frac{-2}{2} = -1$$

Linear System of Equations

How do we solve $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ when $\boldsymbol{A}$ is a non-triangular matrix?

We can perform LU factorization: given a $n \times n$ matrix $\boldsymbol{A}$, obtain lower triangular matrix $\boldsymbol{L}$ and upper triangular matrix $\boldsymbol{U}$ such that

$$\boldsymbol{A} = \boldsymbol{LU}$$

where we set the diagonal entries of $\boldsymbol{L}$ to be equal to 1.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ L_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{pmatrix} = \overbrace{\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}}^{A}$$

1. $\underset{\approx}{A} = \underset{\approx}{L}\,\underset{\approx}{U}$ ✓

$$Ax = b \longrightarrow \underset{\approx}{L}\,\boxed{\underset{\approx}{U}\,\underset{\sim}{x}} = \underset{\sim}{b}$$

$$\underset{\sim}{y}$$

2. $\underset{\approx}{L}\,\boxed{\underset{\sim}{y}} = \underset{\sim}{b} \longrightarrow$ solve for $\underset{\sim}{y}$

Forward.

3. $\underset{\approx}{U}\,\boxed{\underset{\sim}{x}} = \underset{\sim}{y}$

$\longrightarrow$ solve for $\underset{\sim}{x}$ (Backward) la.lu(A) = P, L, U

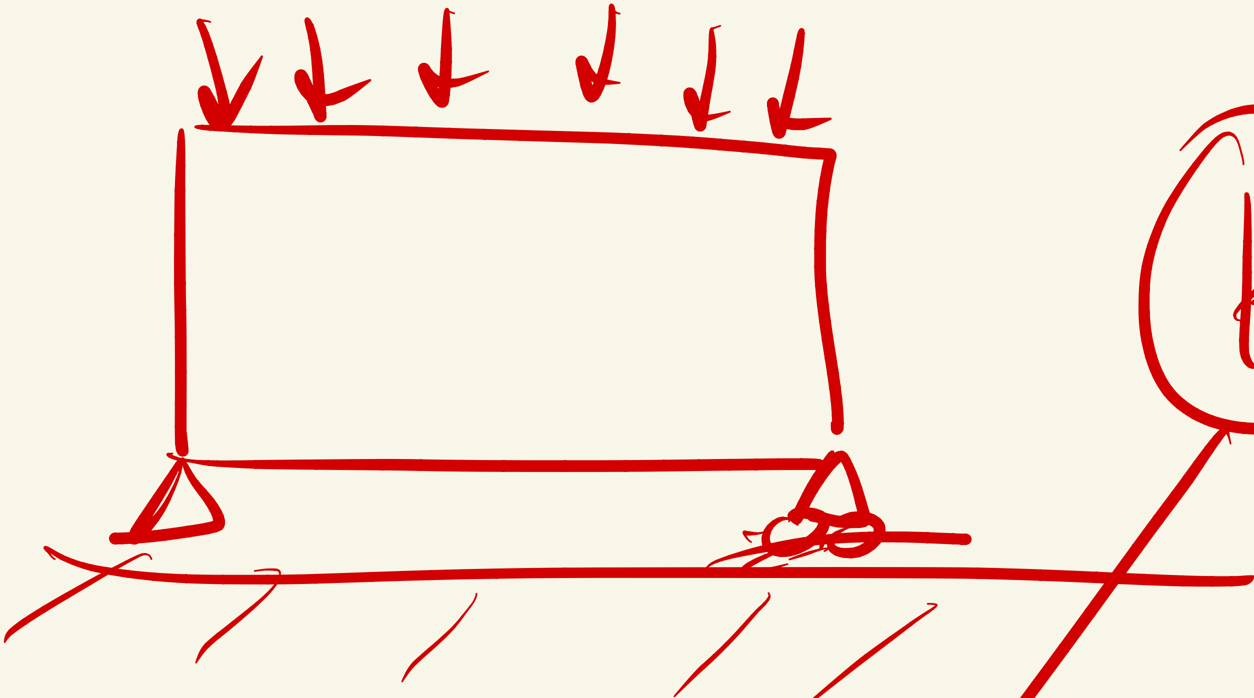
$$A = PLU$$

$$Ax = b$$

$$PLUx = b$$

$PLy = b \longrightarrow Ly = P^T b \rightarrow$ solve for y

$Ux = y \longrightarrow$ solve for x



$KU = F$

output

input

input

solve(K, F)

$KU = F$

$\rightarrow$ solve(K, F)

$\rightarrow$ lu$(K) = P, L, U$

Forward

Backward

$K = PLU$

sym

pos. def

$\rightarrow$ Cholesky

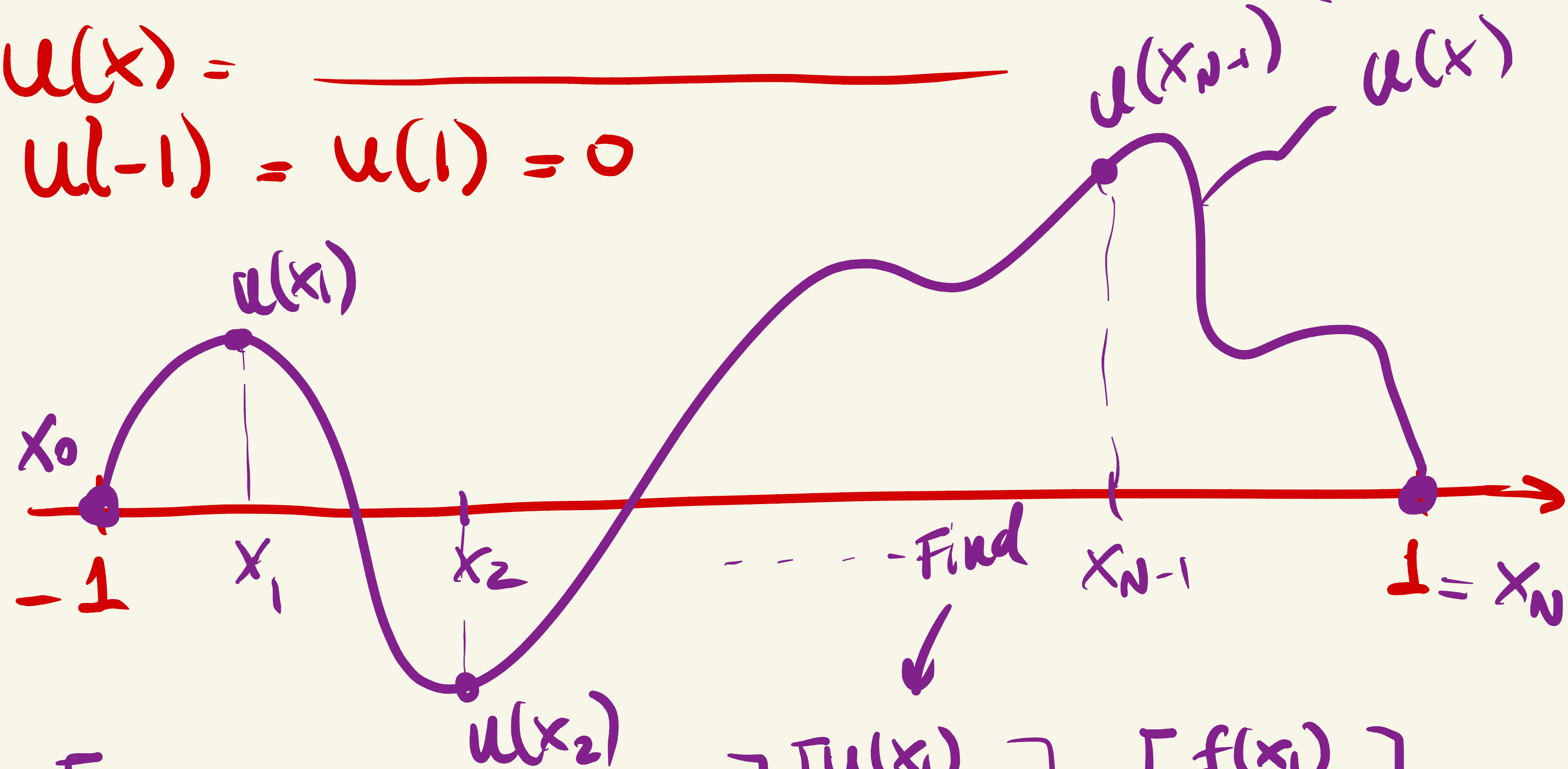
$K = LL^T$

$\rightarrow$ sparse

ODE $\quad -u''(x) + u'(x) = f(x)$

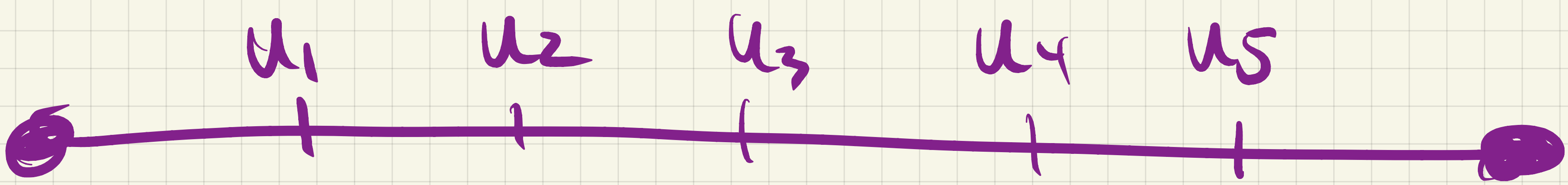
$u(x) = $ ______________________

$u(-1) = u(1) = 0$



$$\begin{bmatrix} & & \\ & \underline{\underline{A}} & \\ & & \end{bmatrix} \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ u \\ \\ \end{bmatrix} \begin{array}{l} \rightarrow x_1 \\ \\ \rightarrow x_{N-1} \end{array} = \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$



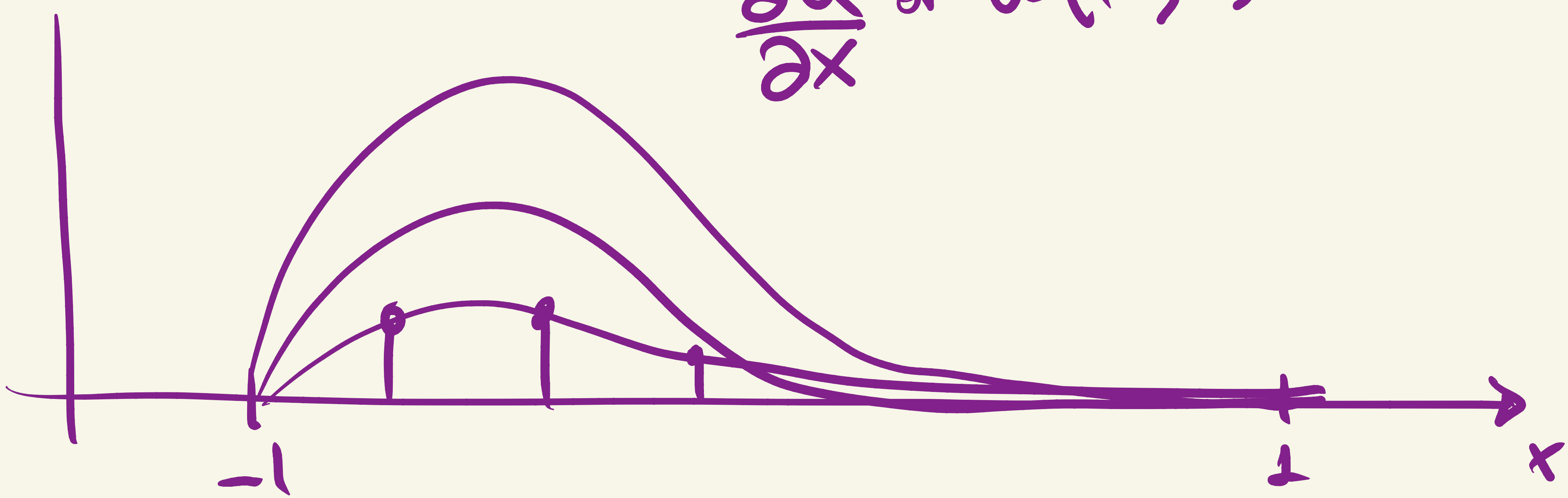
$N = 6$

PDE : $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$

$u(x,t) = ?$

$u(-1,t) = 0$

$\frac{\partial u}{\partial x}$ or $u'(+1,t) = 0$



1000 timesteps

$$A u^{(0)} = b^{(0)} \longrightarrow \text{solve}(A, b^{(0)})$$

$$A u^{(1)} = b^{(1)} \longrightarrow \text{solve}(A, b^{(1)})$$

$$\vdots$$

$$A u^{(t)} = b^{(t)}$$

$$\vdots$$

$$A u^{(n)} = b^{(n)}$$

Cholesky

1. $A = L L^T$ L: lower triang

2. $L \underbrace{L^T x}_{y} = b$

   $\boxed{Ly = b}$ → solve for y

3. $L^T x = y$ → solve for x