Floating point representation

(Unsigned) Fixed-point representation

The numbers are stored with a fixed number of bits for the integer part and a fixed number of bits for the fractional part.

Suppose we have 8 bits to store a real number, where 5 bits store the integer part and 3 bits store the fractional part:

$$\left(\underset{2^4}{1}\ \underset{2^3}{0}\ \underset{2^2}{1}\ \underset{2^1}{1}\ \underset{2^0}{1}.\underset{2^{-1}}{0}\ \underset{2^{-2}}{1}\ \underset{2^{-3}}{1}\right)_2$$

<u>Smallest number:</u>

<u>Largest number:</u>

(Unsigned) Fixed-point representation

Suppose we have 64 bits to store a real number, where 32 bits store the integer part and 32 bits store the fractional part:

$$(a_{31} \ldots a_2 a_1 a_0 . b_1 b_2 b_3 \ldots b_{32})_2 = \sum_{k=0}^{31} a_k \, 2^k + \sum_{k=1}^{32} b_k \, 2^{-k}$$

$$= a_{31} \times 2^{31} + a_{30} \times 2^{30} + \cdots + a_0 \times 2^0 + b_1 \times 2^{-1} + b_2 \times 2^{2} + \cdots + b_{32} \times 2^{-32}$$

0 ∞

(Unsigned) Fixed-point representation

Range: difference between the largest and smallest numbers possible.

More bits for the integer part $\longrightarrow$ increase range

Precision: smallest possible difference between any two numbers

More bits for the fractional part $\longrightarrow$ increase precision

$(a_2 a_1 a_0 . b_1 b_2 b_3)_2$ OR $(a_1 a_0 . b_1 b_2 b_3 b_4)_2$

Wherever we put the binary point, there is a trade-off between the amount of range and precision. **It can be hard to decide how much you need of each!**

Scientific Notation

In **scientific notation**, a number can be expressed in the form

$$x = \pm r \times 10^m$$

where r is a coefficient in the range $1 \leq r < 10$ and m is the exponent.

$1165.7 =$

$0.0004728 =$

Floating-point numbers

A floating-point number can represent numbers of different order of magnitude (very large and very small) with the same number of fixed bits.

In general, in the binary system, a floating number can be expressed as

$$x = \pm\, q \times 2^{m}$$

q is the significand, normally a fractional value in the range $[1.0, 2.0)$

m is the exponent

Floating-point numbers

Numerical Form:

$$x = \pm q \times 2^m = \pm b_0.\underbrace{b_1 b_2 b_3 \ldots b_n}_{\substack{\text{Fractional part of significand} \\ (n \text{ bits})}} \times 2^m$$

Normalized floating-point numbers

Normalized floating point numbers are expressed as

$$x = \pm 1.b_1 b_2 b_3 \ldots b_n \times 2^m = \pm 1.f \times 2^m$$

where f is the fractional part of the significand, m is the exponent and $b_i \in \{0,1\}$.

Converting floating points

Convert $(39.6875)_{10} = (100111.1011)_2$ into floating point representation

Iclicker question

Determine the normalized floating point representation $1.\boldsymbol{f} \times 2^{\boldsymbol{m}}$ of the decimal number $x = 47.125$ ($\boldsymbol{f}$ in binary representation and $\boldsymbol{m}$ in decimal)

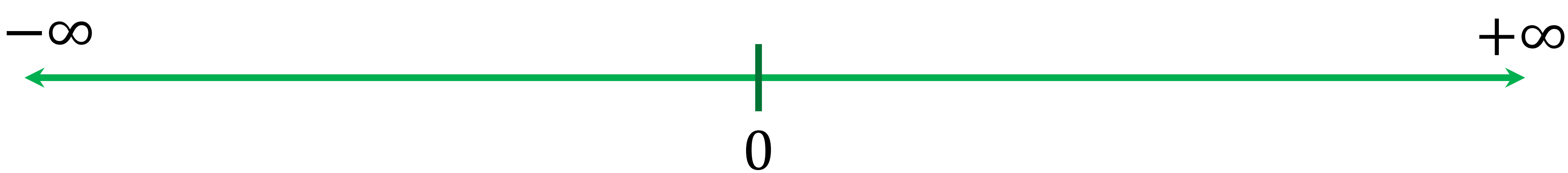
A) $(1.01110001)_2 \times 2^{\mathbf{5}}$
B) $(1.01110001)_2 \times 2^{\mathbf{4}}$
C) $(1.01111001)_2 \times 2^{\mathbf{5}}$
D) $(1.01111001)_2 \times 2^{\mathbf{4}}$

Normalized floating-point numbers

$$x = \pm\, q \times 2^m = \pm\, 1.b_1 b_2 b_3 \ldots b_n \times 2^m = \pm\, 1.f \times\; 2^m$$

- **Exponent range**:
- **Precision**:
- **Smallest positive normalized FP number:**
- **Largest positive normalized FP number:**

Normalized floating point number scale



Floating-point numbers: Simple example

A "toy" number system can be represented as $x = \pm 1.b_1 b_2 \times 2^m$ for $m \in [-4,4]$ and $b_i \in \{0,1\}$.

Floating-point numbers: Simple example

A "toy" number system can be represented as $x = \pm 1.b_1 b_2 \times 2^m$ for $m \in [-4,4]$ and $b_i \in \{0,1\}$.

$(1.00)_2 \times 2^0 = 1$
$(1.01)_2 \times 2^0 = 1.25$
$(1.10)_2 \times 2^0 = 1.5$
$(1.11)_2 \times 2^0 = 1.75$

$(1.00)_2 \times 2^1 = 2$
$(1.01)_2 \times 2^1 = 2.5$
$(1.10)_2 \times 2^1 = 3.0$
$(1.11)_2 \times 2^1 = 3.5$

$(1.00)_2 \times 2^2 = 4.0$
$(1.01)_2 \times 2^2 = 5.0$
$(1.10)_2 \times 2^2 = 6.0$
$(1.11)_2 \times 2^2 = 7.0$

$(1.00)_2 \times 2^3 = 8.0$
$(1.01)_2 \times 2^3 = 10.0$
$(1.10)_2 \times 2^3 = 12.0$
$(1.11)_2 \times 2^3 = 14.0$

$(1.00)_2 \times 2^4 = 16.0$
$(1.01)_2 \times 2^4 = 20.0$
$(1.10)_2 \times 2^4 = 24.0$
$(1.11)_2 \times 2^4 = 28.0$

$(1.00)_2 \times 2^{-1} = 0.5$
$(1.01)_2 \times 2^{-1} = 0.625$
$(1.10)_2 \times 2^{-1} = 0.75$
$(1.11)_2 \times 2^{-1} = 0.875$

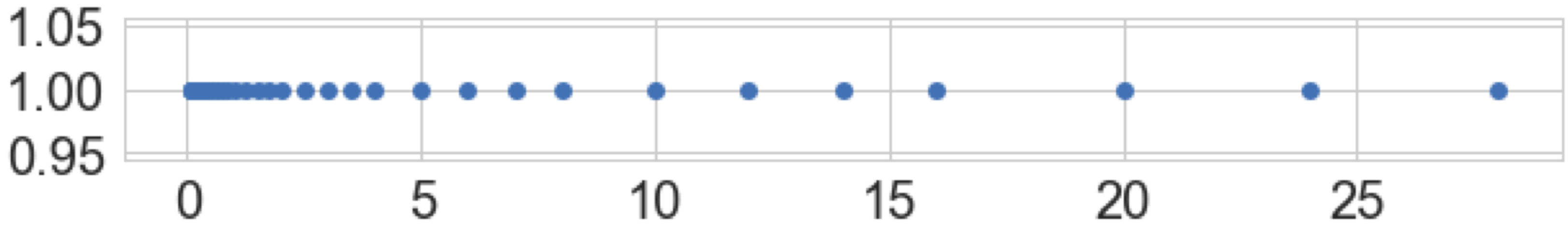
$(1.00)_2 \times 2^{-2} = 0.25$
$(1.01)_2 \times 2^{-2} = 0.3125$
$(1.10)_2 \times 2^{-2} = 0.375$
$(1.11)_2 \times 2^{-2} = 0.4375$

$(1.00)_2 \times 2^{-3} = 0.125$
$(1.01)_2 \times 2^{-3} = 0.15625$
$(1.10)_2 \times 2^{-3} = 0.1875$
$(1.11)_2 \times 2^{-3} = 0.21875$

$(1.00)_2 \times 2^{-4} = 0.0625$
$(1.01)_2 \times 2^{-4} = 0.078125$
$(1.10)_2 \times 2^{-4} = 0.09375$
$(1.11)_2 \times 2^{-4} = 0.109375$

Same steps are performed to obtain the negative numbers. For simplicity, we will show only the positive numbers in this example.

$$x = \pm 1.b_1 b_2 \times 2^m \quad \text{for } m \in [-4,4] \text{ and } b_i \in \{0,1\}$$



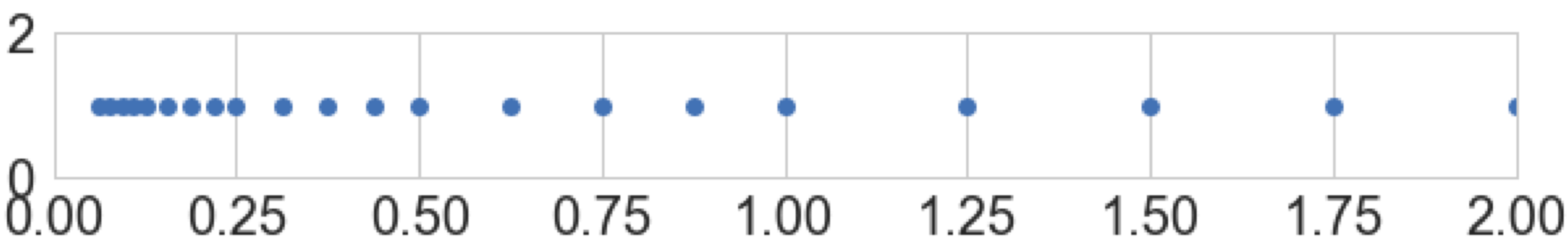
- Smallest normalized positive number:

- Largest normalized positive number:

Machine epsilon

- **Machine epsilon** (ϵ_m): is defined as the distance (gap) between 1 and the next largest floating point number.

$$x = \pm 1.b_1 b_2 \times 2^m \quad \text{for } m \in [-4,4] \text{ and } b_i \in \{0,1\}$$



Machine numbers: how floating point numbers are stored?

Floating-point number representation

What do we need to store when representing floating point numbers in a computer?

$$x = \pm 1.f \times 2^{m}$$

$$x = \underbrace{\boxed{\pm}}_{\text{sign}}\underbrace{\boxed{\quad m \quad}}_{\text{exponent}}\underbrace{\boxed{\qquad f \qquad}}_{\text{significand}}$$

Initially, different floating-point representations were used in computers, generating inconsistent program behavior across different machines.

Around 1980s, computer manufacturers started adopting a standard representation for floating-point number: IEEE (Institute of Electrical and Electronics Engineers) 754 Standard.

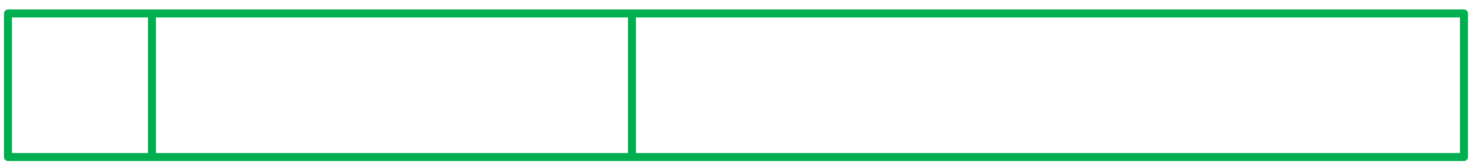
Floating-point number representation

Numerical form:

$$x = \pm 1.f \times 2^{m}$$

Representation in memory:

$x =$

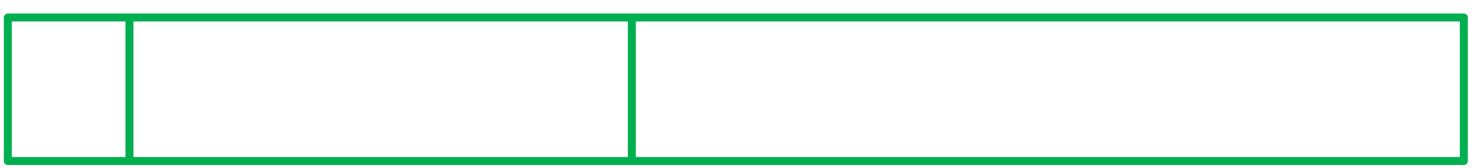


Precisions:

IEEE-754 Single precision (32 bits):

$x =$ 

IEEE-754 Double precision (64 bits):

$x =$ 

IEEE-754 Single Precision (32-bit)

$$x = (-1)^{s}\, 1.f \times 2^{m}$$

s	$c = m + shift$	f
sign (1-bit)	exponent (8-bit)	significand (23-bit)

IEEE-754 Single Precision (32-bit)

$$x = (-1)^{s}\, 1.f \times 2^{m}$$

Example: Represent the number $x = -67.125$ using IEEE Single-Precision Standard

$$67.125 = (1000011.001)_2 = (1.000011001)_2 \times 2^6$$

IEEE-754 Single Precision (32-bit)

$$x = (-1)^{s}\, 1.f \times 2^{m} = \boxed{\;s\;\big|\quad c\quad\big|\qquad f\qquad} \qquad c = m + 127$$

- **Machine epsilon** (ϵ_m): is defined as the distance (gap) between 1 and the next largest floating point number.

- **Smallest positive normalized FP number:**

- **Largest positive normalized FP number:**

Special Values:

$$x = (-1)^{s}\, 1.f \times 2^{m} =$$

s	c	f

1) **Zero**:

$x =$

2) **Infinity**: $+\infty$ $(s = 0)$ and $-\infty$ $(s = 1)$

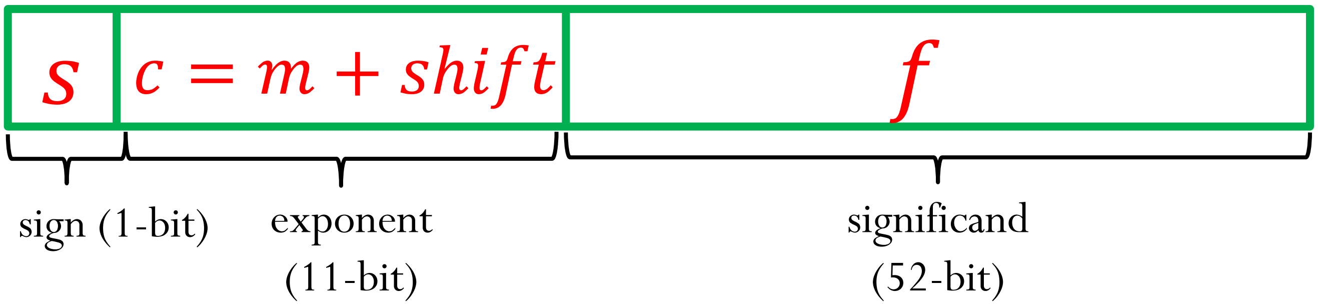
$x =$

3) **NaN**: (results from operations with undefined results)

$x =$

IEEE-754 Double Precision (64-bit)

$$x = (-1)^{\boldsymbol{s}}\, 1.\boldsymbol{f} \times 2^{\boldsymbol{m}}$$



IEEE-754 Double Precision (64-bit)

$$x = (-1)^{s}\, 1.f \times 2^{m} = \boxed{s \mid c \mid f} \qquad c = m + 1023$$

- **Machine epsilon** (ϵ_m): is defined as the distance (gap) between 1 and the next largest floating point number.

- **Smallest positive normalized FP number:**

- **Largest positive normalized FP number:**

Subnormal (or denormalized) numbers

Subnormal (or denormalized) numbers

IEEE-754 Single precision (32 bits):

$c = (00000000)_2 = 0$

IEEE-754 Double precision (64 bits):

$c = (00000000000)_2 = 0$

Subnormal (or denormalized) numbers

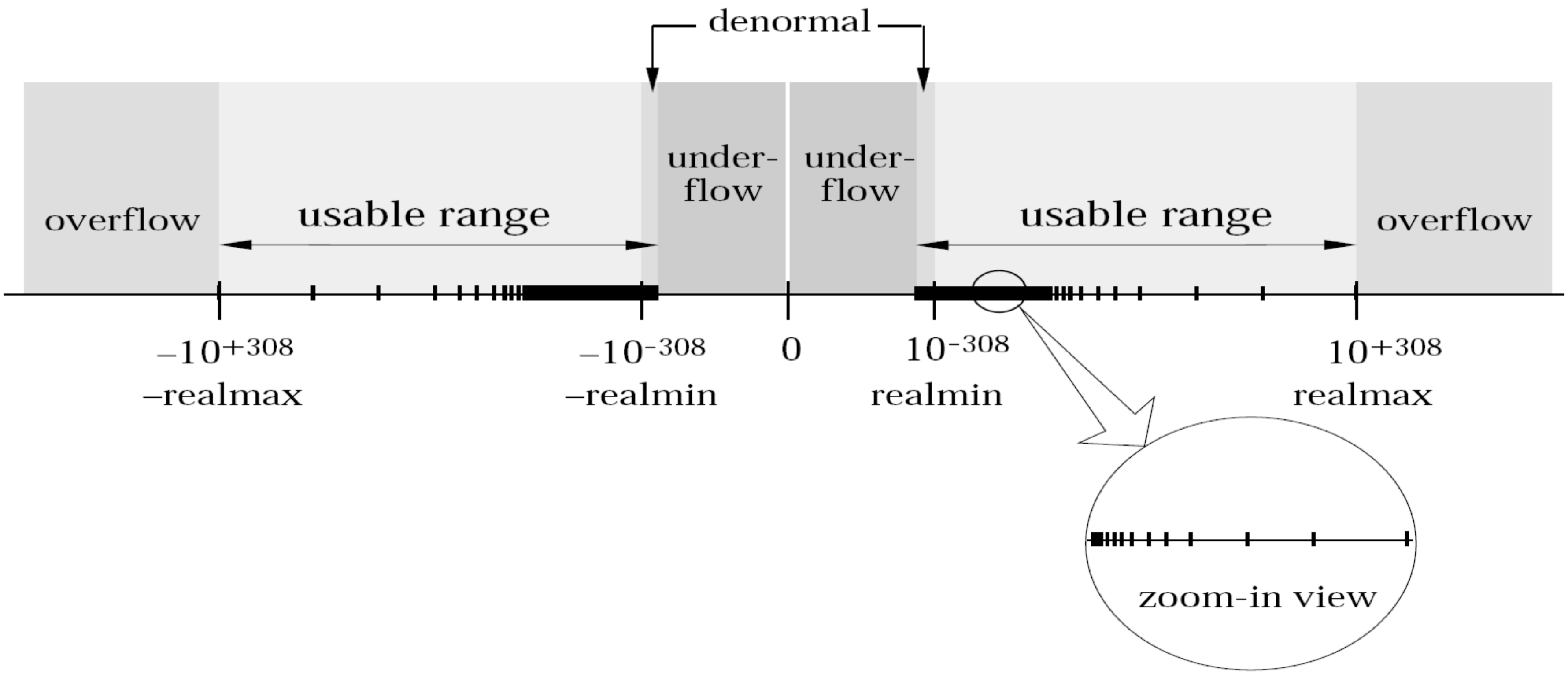
- Noticeable gap around zero, present in any floating system, due to normalization
- Relax the requirement of normalization, and allow the leading digit to be zero, only when the exponent is at its minimum ($m = L$)
- Computations with subnormal numbers are often slow.

Representation in memory (another special case):

$x =$ 

Numerical value:

IEEE-754 Double Precision



Summary for Single Precision

$$x = (-1)^{s}\, 1.f \times 2^{m} = \boxed{s \mid c \mid f} \qquad m = c - 127$$

Stored binary exponent (c)	Significand fraction (f)	value
00000000	0000…0000	zero
00000000	$any\ f \neq 0$	$(-1)^{s}\, 0.f \times 2^{-126}$
00000001	$any\ f$	$(-1)^{s}\, 1.f \times 2^{-126}$
$\vdots$	$\vdots$	$\vdots$
11111110	$any\ f$	$(-1)^{s}\, 1.f \times 2^{127}$
11111111	$any\ f \neq 0$	NaN
11111111	0000…0000	infinity

Iclicker question

A number system can be represented as $x = \pm 1.b_1 b_2 b_3 \times 2^m$ for $m \in [-5,5]$ and $b_i \in \{0,1\}$.

1) **What is the smallest positive normalized FP number:**
   a) 0.0625 b) 0.09375 c) 0.03125 d) 0.046875 e) 0.125

2) **What is the largest positive normalized FP number:**
   a) 28 b) 60 c) 56 d) 32

3) **How many additional numbers (positive and negative) can be represented when using subnormal representation?**
   a) 7 b) 14 c) 3 d) 6 e) 16

4) **What is the smallest positive subnormal number?**
   a) 0.00390625 b) 0.00195313 c) 0.03125 d) 0.0136719

5) **Determine machine epsilon**
   a) 0.0625 b) 0.00390625 c) 0.0117188 d) 0.125

A number system can be represented as $x = \pm 1.b_1 b_2 b_3 b_4 \times 2^m$ for $m \in [-6,6]$ and $b_i \in \{0,1\}$.

1) Let's say you want to represent the decimal number 19.625 using the binary number system above. Can you represent this number exactly?

2) What is the range of integer numbers that you can represent exactly using this binary system?