Solving Linear Least Squares with SVD

What we have learned so far…

$\boldsymbol{A}$ is a $m \times n$ matrix where $m > n$
(more points to fit than coefficient to be determined)

Normal Equations: $\boldsymbol{A}^T \boldsymbol{A}\, \boldsymbol{x} = \boldsymbol{A}^T\, \boldsymbol{b}$

- The solution $\boldsymbol{A}\, \boldsymbol{x} \cong \boldsymbol{b}$ is unique if and only if $rank(\mathbf{A}) = n$

($\boldsymbol{A}$ is full column rank)

- $rank(\mathbf{A}) = n$ $\rightarrow$ columns of $\boldsymbol{A}$ are ***linearly independent*** $\rightarrow$ n non-zero singular values $\rightarrow$ $\boldsymbol{A}^T\, \boldsymbol{A}$ has only positive eigenvalues $\rightarrow$ $\boldsymbol{A}^T \boldsymbol{A}$ is a symmetric and positive definite matrix $\rightarrow$ $\boldsymbol{A}^T \boldsymbol{A}$ is invertible

$$\boldsymbol{x} = (\boldsymbol{A}^T \boldsymbol{A})^{-1} \boldsymbol{A}^T\, \boldsymbol{b}$$

- If $rank(\mathbf{A}) < n$, then $\boldsymbol{A}$ is rank-deficient, and solution of linear least squares problem is ***not unique***.

SVD to solve linear least squares problems

$\boldsymbol{A}$ is a $m \times n$ rectangular matrix where $m > n$, and hence the SVD decomposition is given by:

$$\boldsymbol{A} = \begin{pmatrix} \vdots & \cdots & \vdots \\ \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_m \\ \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & 0 \\ & & \vdots \\ & & 0 \end{pmatrix} \begin{pmatrix} \cdots & \mathbf{v}_1^T & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \mathbf{v}_n^T & \cdots \end{pmatrix}$$

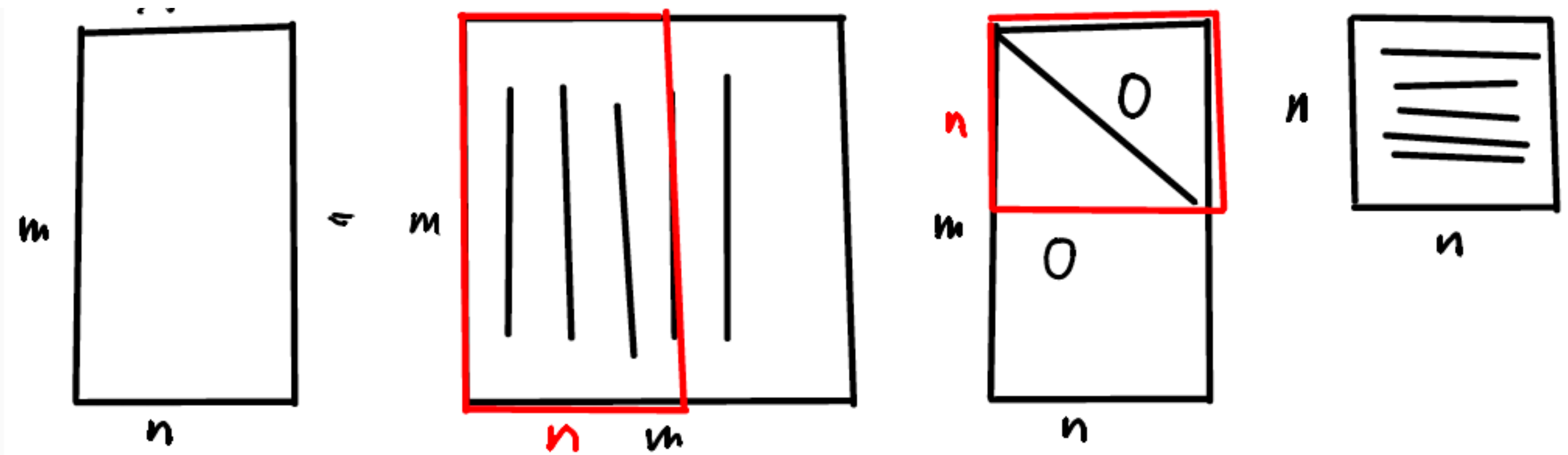
We want to find the least square solution of $\boldsymbol{A}\,\boldsymbol{x} \cong \boldsymbol{b}$, where $\boldsymbol{A} = \boldsymbol{U}\,\boldsymbol{\Sigma}\,\boldsymbol{V}^{\boldsymbol{T}}$

or better expressed in reduced form: $\boldsymbol{A} = \boldsymbol{U}_R\ \boldsymbol{\Sigma}_{\boldsymbol{R}}\ \boldsymbol{V}^{\boldsymbol{T}}$

Recall Reduced SVD $m > n$

$$A = U_R \; \Sigma_R \; V^T$$

A: $m \times n$, U_R: $m \times n$, Σ_R: $n \times n$, V^T: $n \times n$



Shapes of the Reduced SVD

Suppose you compute a reduced SVD $A = U\Sigma V^T$ of a 10×14 matrix A. What will the shapes of U, Σ, and V be?

Hint: Remember the transpose on V!

The shape of U will be ____ $\times$ ____ .

The shape of Σ will be ____ $\times$ ____ .

The shape of V will be ____ $\times$ ____ .

SVD to solve linear least squares problems

$$\boldsymbol{A} = \boldsymbol{U}_R \ \boldsymbol{\Sigma}_R \ \boldsymbol{V}^T$$

$$\boldsymbol{A} = \begin{pmatrix} \vdots & \cdots & \vdots \\ \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_n \\ \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \begin{pmatrix} \cdots & \mathbf{v}_1^T & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \mathbf{v}_n^T & \cdots \end{pmatrix}$$

Example:

Consider solving the least squares problem $\boldsymbol{A}\,\boldsymbol{x} \cong \boldsymbol{b}$, where the singular value decomposition of the matrix $\boldsymbol{A} = \boldsymbol{U}\,\boldsymbol{\Sigma}\,\boldsymbol{V}^{\boldsymbol{T}}\boldsymbol{x}$ is:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} \cong \begin{bmatrix} 12 \\ 9 \\ 9 \\ 10 \end{bmatrix}$$

Determine $\|\boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}\|_2$

Example

Suppose you have $\boldsymbol{A} = \boldsymbol{U}\,\boldsymbol{\Sigma}\,\boldsymbol{V}^{\boldsymbol{T}}\boldsymbol{x}$ calculated. What is the cost of solving

$$\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}\|_2^2\ ?$$

A) $O(n)$
B) $O(n^2)$
C) $O(mn)$
D) $O(m)$
E) $O(m^2)$