Eigenvalues and Eigenvectors

Eigenvalue problem

Let $\boldsymbol{A}$ be an $n \times n$ matrix:

$\boldsymbol{x} \neq \boldsymbol{0}$ is an **eigenvector** of $\boldsymbol{A}$ if there exists a scalar λ such that

$$\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$$

where λ is called an **eigenvalue**.

If $\boldsymbol{x}$ is an eigenvector, then $\alpha\boldsymbol{x}$ is also an eigenvector. Therefore, we will usually seek for **normalized eigenvectors**, so that

$$\|\boldsymbol{x}\| = 1$$

Note: When using Python, numpy.linalg.eig will normalize using $p=2$ norm.

How do we find eigenvalues?

Linear algebra approach:

$$\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$$
$$(\boldsymbol{A} - \lambda\,\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

Therefore the matrix $(\boldsymbol{A} - \lambda\,\boldsymbol{I})$ is singular $\Longrightarrow det(\boldsymbol{A} - \lambda\,\boldsymbol{I}) = 0$

$p(\lambda) = \ det(\boldsymbol{A} - \lambda\,\boldsymbol{I})$ is the characteristic polynomial of degree n.

In most cases, there is no analytical formula for the eigenvalues of a matrix (Abel proved in 1824 that there can be no formula for the roots of a polynomial of degree 5 or higher) $\Longrightarrow$ **Approximate the eigenvalues numerically**!

Example

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \qquad det\begin{pmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{pmatrix} = 0$$

Diagonalizable Matrices

A $n \times n$ matrix $\boldsymbol{A}$ with n linearly independent eigenvectors $\boldsymbol{u}$ is said to be **diagonalizable**.

$$\boldsymbol{A}\,\boldsymbol{u_1} = \lambda_1\,\boldsymbol{u_1},$$
$$\boldsymbol{A}\,\boldsymbol{u_2} = \lambda_2\,\boldsymbol{u_2},$$
$$\ldots$$
$$\boldsymbol{A}\,\boldsymbol{u_n} = \lambda_n\,\boldsymbol{u_n},$$

Example

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \quad det\begin{pmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{pmatrix} = 0$$

Solution of characteristic polynomial gives: $\lambda_1 = 4, \lambda_2 = 0$

To get the eigenvectors, we solve: $\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$

$$\begin{pmatrix} 2-(4) & 1 \\ 4 & 2-(4) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2-(0) & 1 \\ 4 & 2-(0) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Example

The eigenvalues of the matrix:

$$\boldsymbol{A} = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$$

are $\lambda_1 = \lambda_2 = -3$.

Select the **incorrect** statement:

A) Matrix $\boldsymbol{A}$ is diagonalizable
B) The matrix $\boldsymbol{A}$ has only one eigenvalue with multiplicity 2
C) Matrix $\boldsymbol{A}$ has only one linearly independent eigenvector
D) Matrix $\boldsymbol{A}$ is not singular

Let's look back at diagonalization…

1) If a $n \times n$ matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{x}$ then $\boldsymbol{A}$ is diagonalizable, i.e.,

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$$

where the columns of $\boldsymbol{U}$ are the linearly independent normalized eigenvectors $\boldsymbol{x}$ of $\boldsymbol{A}$ (which guarantees that $\boldsymbol{U}^{-1}$ exists) and $\boldsymbol{D}$ is a diagonal matrix with the eigenvalues of $\boldsymbol{A}$.

2) If a $n \times n$ matrix $\boldsymbol{A}$ has less then n linearly independent eigenvectors, the matrix is called defective (and therefore not diagonalizable).

3) If a $n \times n$ **symmetric** matrix $\boldsymbol{A}$ has n distinct eigenvalues then $\boldsymbol{A}$ is diagonalizable.

A $\boldsymbol{n}\times\boldsymbol{n}$ symmetric matrix $\boldsymbol{A}$ with $\boldsymbol{n}$ distinct eigenvalues is diagonalizable.

Suppose $\lambda, \boldsymbol{u}$ and $\mu, \boldsymbol{v}$ are eigenpairs of $\boldsymbol{A}$

$$\lambda\,\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u}$$
$$\mu\,\boldsymbol{v} = \boldsymbol{A}\boldsymbol{v}$$

Some things to remember about eigenvalues:

- Eigenvalues can have zero value
- Eigenvalues can be negative
- Eigenvalues can be real or complex numbers
- A $n \times n$ real matrix can have complex eigenvalues
- The eigenvalues of a $n \times n$ matrix are not necessarily unique. In fact, we can define the multiplicity of an eigenvalue.
- If a $n \times n$ matrix has n linearly independent eigenvectors, then the matrix is diagonalizable

How can we get eigenvalues numerically?

Assume that $\boldsymbol{A}$ is diagonalizable (i.e., it has n linearly independent eigenvectors $\boldsymbol{u}$). We can propose a vector $\boldsymbol{x}$ which is a linear combination of these eigenvectors:

$$\boldsymbol{x} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n$$

Power Iteration

Our goal is to find an eigenvector $\boldsymbol{u}_i$ of $\boldsymbol{A}$. We will use an iterative process, where we start with an initial vector, where here we assume that it can be written as a linear combination of the eigenvectors of $\boldsymbol{A}$.

$$\boldsymbol{x}_0 = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n$$

Power Iteration

$$\boldsymbol{x}_k = (\lambda_1)^k \left[\alpha_1 \boldsymbol{u}_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k \boldsymbol{u}_2 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k \boldsymbol{u}_n\right]$$

Assume that $\alpha_1 \neq 0$, the term $\alpha_1 \boldsymbol{u}_1$ dominates the others when k is very large.

Since $|\lambda_1| > |\lambda_2|$, we have $\left(\frac{\lambda_2}{\lambda_1}\right)^k \ll 1$ when k is large

Hence, as k increases, $\boldsymbol{x}_k$ converges to a multiple of the first eigenvector $\boldsymbol{u}_1$, i.e.,