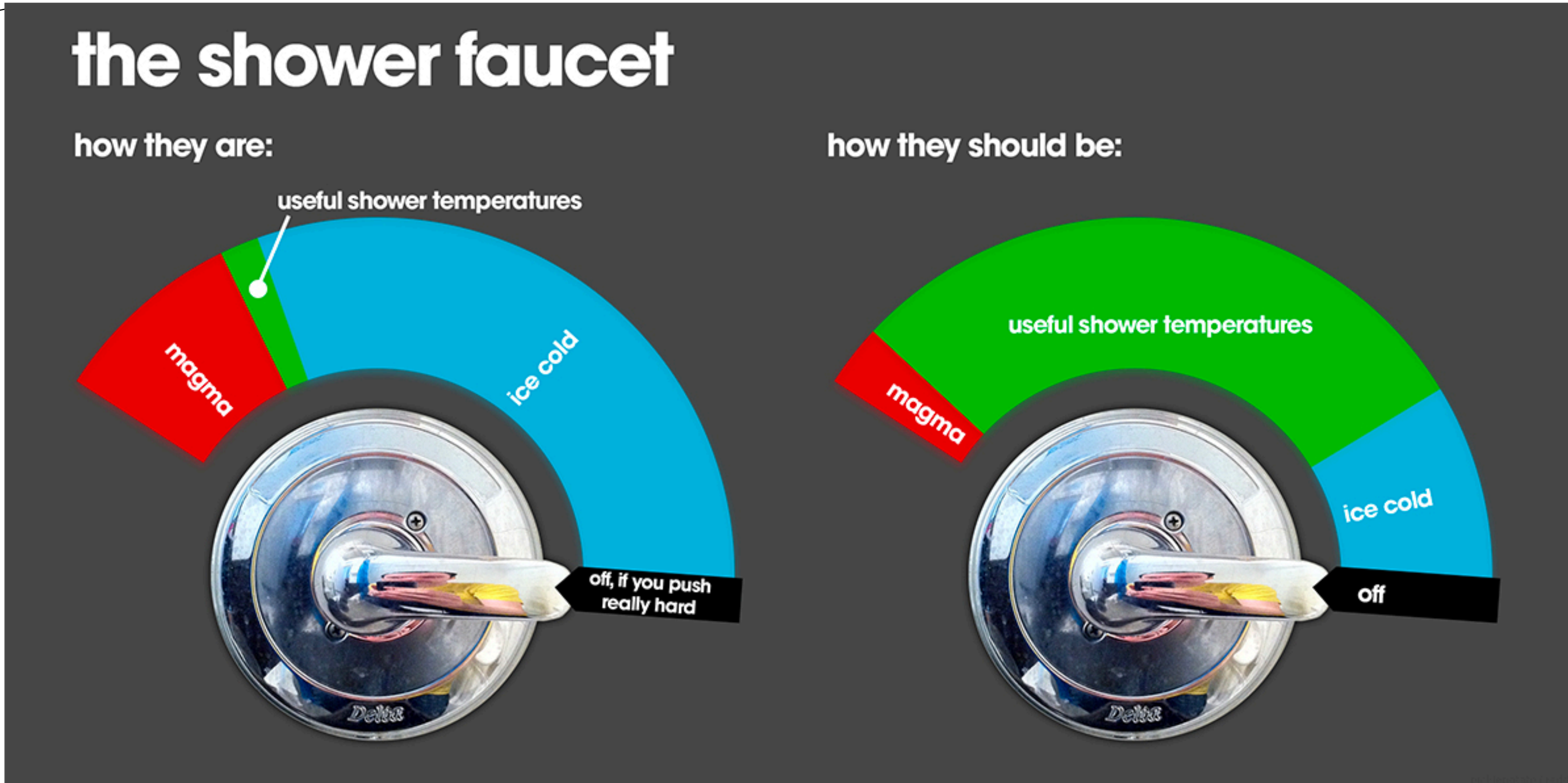
Linear System of Equations - Conditioning



the shower faucet
how they are:
useful shower temperatures
magma
ice cold
off, if you push really hard
how they should be:
useful shower temperatures
magma
ice cold
off
Delta
Delta

WHAT IT LOOKS LIKE
WHAT IT FEELS LIKE

Numerical experiments

Input has uncertainties:

- Errors due to representation with finite precision
- Error in the sampling

Once you select your numerical method , how much error should you expect to see in your **output?**

Is your method sensitive to errors (perturbation) in the input?

Demo “HilbertMatrix-ConditionNumber”

Sensitivity of Solutions of Linear Systems

Suppose we start with a non-singular system of linear equations $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$.

We change the right-hand side vector $\boldsymbol{b}$ (input) by a small amount $\Delta\boldsymbol{b}$.

How much the solution $\boldsymbol{x}$ (output) changes, i.e., how large is $\Delta\boldsymbol{x}$?

$$\frac{\text{Output Relative error}}{\text{Input Relative error}} = \frac{\|\Delta\boldsymbol{x}\|/\|\boldsymbol{x}\|}{\|\Delta\boldsymbol{b}\|/\|\boldsymbol{b}\|} = \frac{\|\Delta\boldsymbol{x}\|\,\|\boldsymbol{b}\|}{\|\Delta\boldsymbol{b}\|\,\|\boldsymbol{x}\|}$$

Sensitivity of Solutions of Linear Systems

$$\frac{\text{Output Relative error}}{\text{Input Relative error}} = \frac{\|\Delta \boldsymbol{x}\| / \|\boldsymbol{x}\|}{\|\Delta \boldsymbol{b}\| / \|\boldsymbol{b}\|} = \frac{\|\Delta \boldsymbol{x}\| \, \|\boldsymbol{b}\|}{\|\Delta \boldsymbol{b}\| \, \|\boldsymbol{x}\|}$$

$$\frac{\text{Output Relative error}}{\text{Input Relative error}} =$$

Sensitivity of Solutions of Linear Systems

$$\frac{\|\Delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \|\boldsymbol{A}^{-1}\| \, \|\boldsymbol{A}\| \, \frac{\|\Delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|}$$

Sensitivity of Solutions of Linear Systems

We can also add a perturbation to the matrix $\boldsymbol{A}$ (input) by a small amount $\boldsymbol{E}$, such that

$$(\boldsymbol{A} + \boldsymbol{E})\,\widehat{\boldsymbol{x}} = \boldsymbol{b}$$

and in a similar way obtain:

$$\frac{\|\Delta \boldsymbol{x}\,\|}{\|\boldsymbol{x}\|} \leq \|\boldsymbol{A}^{-1}\|\,\|\boldsymbol{A}\|\,\frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|}$$

Condition number

The condition number is a measure of sensitivity of solving a linear system of equations to variations in the input.

The condition number of a matrix $\boldsymbol{A}$:

$$cond(\boldsymbol{A}) = \|\boldsymbol{A}^{-1}\|\ \|\boldsymbol{A}\|$$

Recall that the induced matrix norm is given by

$$\|\boldsymbol{A}\| = \max_{\|\boldsymbol{x}\|=1} \|\boldsymbol{A}\boldsymbol{x}\|$$

And since the condition number is relative to a given norm, we should be precise and for example write:

$$cond_2(\boldsymbol{A}) \text{ or } cond_\infty(\boldsymbol{A})$$

Demo “HilbertMatrix-ConditionNumber”

Condition number

$$\frac{\|\Delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq cond(\boldsymbol{A}) \frac{\|\Delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|}$$

Small condition numbers mean not a lot of error amplification. Small condition numbers are good!

But how small?

Condition number

$$\frac{||\Delta \boldsymbol{x}\ ||}{||\boldsymbol{x}||} \leq cond(\boldsymbol{A})\ \frac{||\Delta \boldsymbol{b}||}{||\boldsymbol{b}||}$$

Small condition numbers mean not a lot of error amplification. Small condition numbers are good!

Recall that

$$||\boldsymbol{I}|| = \max_{||\boldsymbol{x}||=1} ||\boldsymbol{I}\ \boldsymbol{x}|| = 1$$

Which provides with a lower bound for the condition number:

$$cond(\boldsymbol{A}) = ||\boldsymbol{A}^{-1}||\ ||\boldsymbol{A}|| \geq ||\boldsymbol{A}^{-1}\boldsymbol{A}|| = ||\boldsymbol{I}|| = 1$$

If $\boldsymbol{A}^{-1}$ does not exist, then $cond(\boldsymbol{A}) = \infty$ (by convention)

Recall Induced Matrix Norms

$$\|\boldsymbol{A}\|_1 = \max_j \sum_{i=1}^{n} |A_{ij}|$$

Maximum absolute column sum of the matrix $\boldsymbol{A}$

$$\|\boldsymbol{A}\|_\infty = \max_i \sum_{j=1}^{n} |A_{ij}|$$

Maximum absolute row sum of the matrix $\boldsymbol{A}$

$$\|\boldsymbol{A}\|_2 = \max_k \sigma_k$$

σ_k are the singular value of the matrix $\boldsymbol{A}$

Condition Number of a Diagonal Matrix

What is the 2-norm-based condition number of the diagonal matrix

$$A = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}?$$

Condition Number of Orthogonal Matrices

What is the 2-norm condition number of an orthogonal matrix A?

$$cond(\boldsymbol{A}) = \|\boldsymbol{A}^{-1}\|_2 \ \|\boldsymbol{A}\|_2 = \left\|\boldsymbol{A}^T\right\|_2 \|\boldsymbol{A}\|_2 = 1$$

That means orthogonal matrices have optimal conditioning.

They are very well-behaved in computation.

About condition numbers

1. For any matrix $\boldsymbol{A}$, $cond(\boldsymbol{A}) \geq 1$
2. For the identity matrix $\boldsymbol{I}$, $cond(\boldsymbol{I}) = 1$
3. For any matrix $\boldsymbol{A}$ and a nonzero scalar γ, $cond(\gamma\boldsymbol{A}) = cond(\boldsymbol{A})$
4. For any diagonal matrix $\boldsymbol{D}$, $cond(\boldsymbol{D}) = \frac{\boldsymbol{max}|d_{\boldsymbol{i}}|}{\boldsymbol{min}|d_{\boldsymbol{i}}|}$
5. The condition number is a measure of how close a matrix is to being singular: a matrix with large condition number is nearly singular, whereas a matrix with a condition number close to 1 is far from being singular
6. The determinant of a matrix is NOT a good indicator is a matrix is near singularity

Residual versus error

Our goal is to find the solution $\boldsymbol{x}$ to the linear system of equations $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$

Let us recall the solution of the perturbed problem

$$\widehat{\boldsymbol{x}} = (\boldsymbol{x} + \Delta\boldsymbol{x})$$

which could be the solution of

$$\boldsymbol{A}\,\widehat{\boldsymbol{x}} = (\boldsymbol{b} + \Delta\boldsymbol{b}), \qquad (\boldsymbol{A} + \boldsymbol{E})\widehat{\boldsymbol{x}} = \boldsymbol{b}, \qquad (\boldsymbol{A} + \boldsymbol{E})\,\widehat{\boldsymbol{x}} = (\boldsymbol{b} + \Delta\boldsymbol{b})$$

And the **error vector** as

$$\boldsymbol{e} = \Delta\boldsymbol{x} = \widehat{\boldsymbol{x}} - \boldsymbol{x}$$

We can write the **residual vector** as

$$\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\,\widehat{\boldsymbol{x}}$$

Relative residual: $\frac{||\boldsymbol{r}||}{||\boldsymbol{A}||\,||\boldsymbol{x}||}$

(How well the solution satisfies the problem)

Relative error: $\frac{||\Delta \boldsymbol{x}\ ||}{||\boldsymbol{x}||}$

(How close the approximated solution is from the exact one)

Residual versus error

It is possible to show that the residual satisfy the following inequality:

$$\frac{||\boldsymbol{r}||}{||\boldsymbol{A}||\,||\widehat{\boldsymbol{x}}||} \leq c\ \epsilon_m$$

Where c is "large" constant when LU/Gaussian elimination is performed without pivoting and "small" with partial pivoting.

Therefore, Gaussian elimination with partial pivoting yields **small relative residual regardless of conditioning of the system**.

When solving a system of linear equations via LU with partial pivoting, the relative residual is guaranteed to be small!

Residual versus error

Let us first obtain the norm of the error:

Rule of thumb for conditioning

Suppose we want to find the solution $\boldsymbol{x}$ to the linear system of equations $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ using LU factorization with partial pivoting and backward/forward substitutions.

Suppose we compute the solution $\widehat{\boldsymbol{x}}$.

If the entries in $\boldsymbol{A}$ and $\boldsymbol{b}$ are accurate to S decimal digits,

and $cond(\boldsymbol{A}) = \mathbf{10}^{\boldsymbol{W}}$,

then the elements of the solution vector $\widehat{\boldsymbol{x}}$ will be accurate to about

$$S - W$$

decimal digits