Truncation errors: using Taylor series to approximation functions

Approximating functions using polynomials:

Let's say we want to approximate a function $f(x)$ with a polynomial

$$f(x) = a_o + a_1\,x + \, a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots$$

For simplicity, assume we know the function value and its derivatives at $x_o = 0$ (we will later generalize this for any point). Hence,

$$f'(x) = a_1 + \, 2\,a_2\,x + 3\,a_3\,x^2 + 4\,a_4\,x^3 + \cdots$$

$$f''(x) = 2\,a_2 + \, (3\times2)a_3\,x + (4\times3)a_4\,x^2 + \cdots$$

$$f'''(x) = (3\times2)a_3 \, + (4\times3\times2)a_4\,x + \cdots$$

$$f'^{v}(x) = (4\times3\times2)a_4 + \cdots$$

$f(0) = a_o$ $\quad$ $f''(0) = 2\,a_2$ $\quad$ $f'^{v}(0) = (4\times3\times2)\,a_4$

$f'(0) = a_1$ $\quad$ $f'''(0) = (3\times2)\,a_3$

$$f^{(i)}(0) = i!\;a_i$$

Taylor Series

Taylor Series approximation about point $x_o = 0$

$$f(x) = a_o + a_1\, x + \; a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

Taylor Series

In a more general form, the Taylor Series approximation about point x_o is given by:

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f'''(x_o)}{3!}(x - x_o)^3 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_o)}{i!}(x - x_o)^i$$

Iclicker question

Assume a finite Taylor series approximation that converges everywhere for a given function $f(x)$ and you are given the following information:

$$f(1) = 2; f'(1) = -3; f''(1) = 4;\ f^{(n)}(1) = 0\ \forall\, n\ \geq 3$$

Evaluate $f(4)$

A) 29
B) 11
C) -25
D) -7
E) None of the above

Taylor Series

We cannot sum infinite number of terms, and therefore we have to **truncate**.

How **big is the error** caused by truncation? Let's write $h = x - x_o$

$$f(x_o + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!}(h)^i = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_o)}{i!}(h)^i$$

And as $h \to 0$ we write:

$$\left| f(x_o + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!}(h)^i \right| \leq C \cdot h^{n+1}$$

Error due to Taylor approximation of degree n

$$\left| f(x_o + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!}(h)^i \right| = O(h^{n+1})$$

Taylor series with remainder

Let f be $(n+1)$-times differentiable on the interval (x_o, x) with $f^{(n)}$ continuous on $[x_o, x]$, and $h = x - x_o$

$$f(x_o + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!}(h)^i = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_o)}{i!}(h)^i$$

Then there exists a $\xi \in (x_o, x)$ so that

$$f(x_o + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!}(h)^i = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(\xi - x_o)^{n+1}}_{\text{Taylor remainder}}$$

$$f(x) - T(x) = R(x)$$

And since $|\xi - x_o| \leq h$

$$f(x_o + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_o)}{i!}(h)^i \leq \frac{f^{(n+1)}(\xi)}{(n+1)!}(h)^{n+1}$$

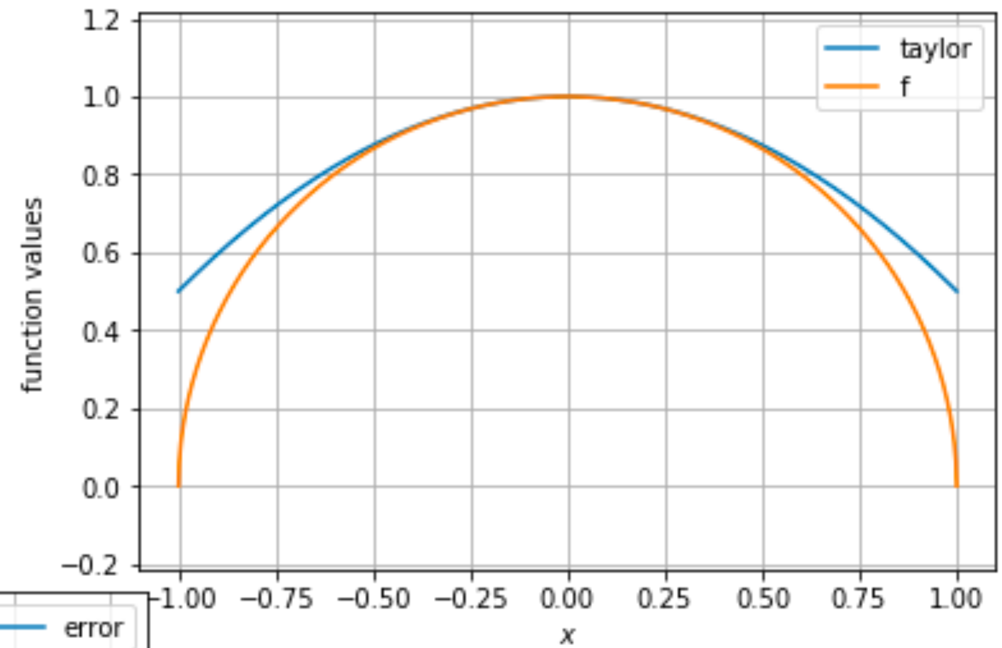
Demo: Polynomial Approximation with Derivatives

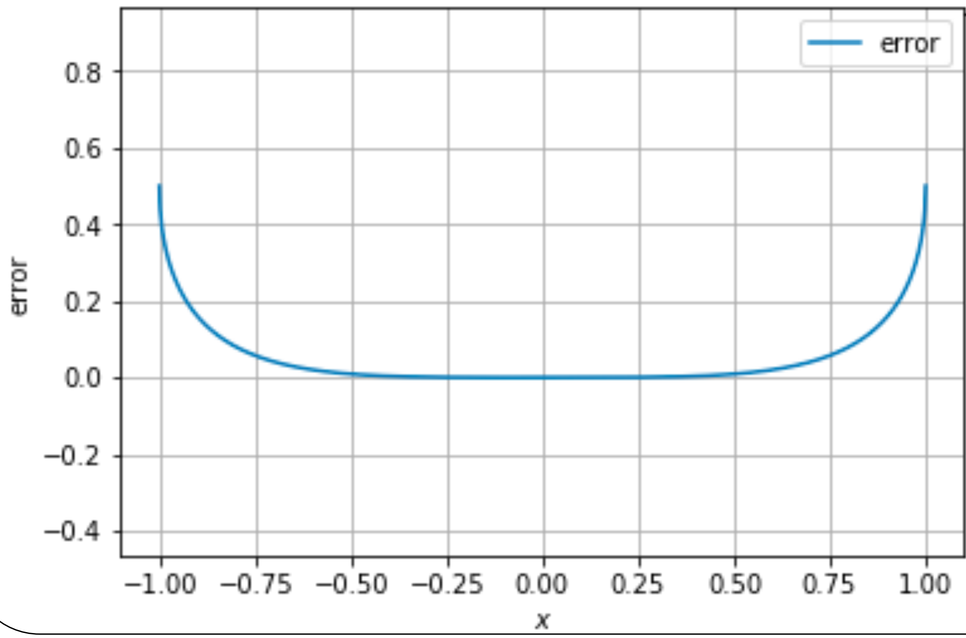
`f`

$$\sqrt{-x^2 + 1}$$

`taylor`

$$-\frac{x^2}{2} + 1$$






```
error = taylor - f
```

Demo: Polynomial Approximation with Derivatives

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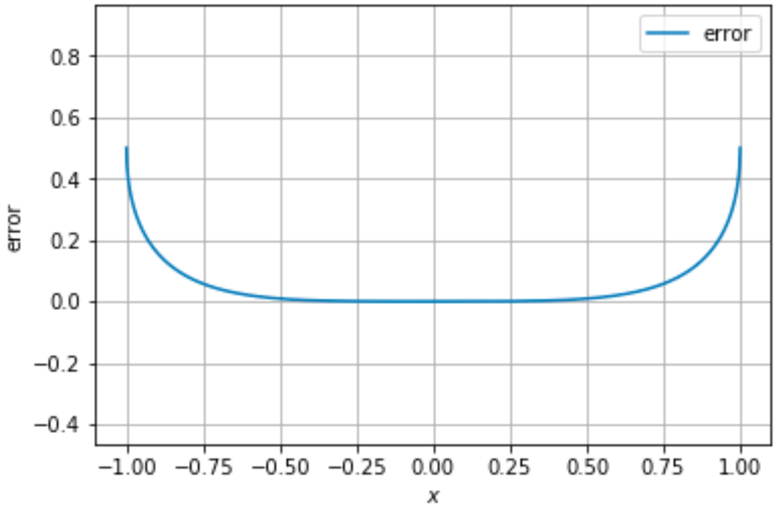
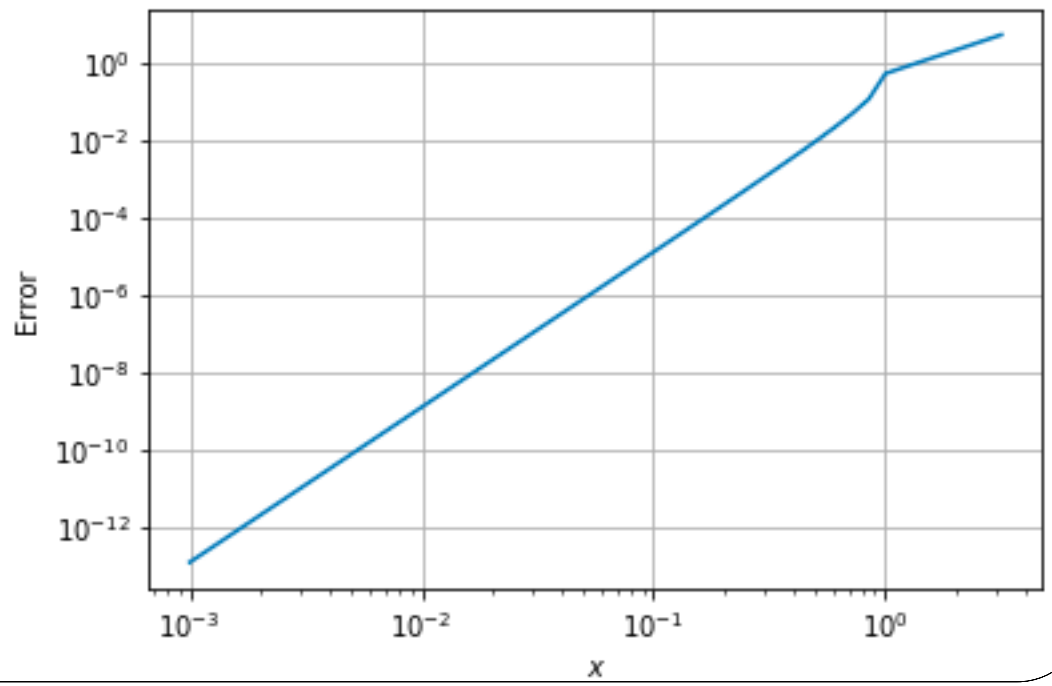




Iclicker question

Error Order for Taylor series

1 point

The series expansion for e^x about 2 is

$$\exp(2) \cdot \left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \ldots\right).$$

If we evaluate e^x using only the first four terms of this expansion (i.e. only terms up to and including $\frac{(x-2)^3}{3!}$), then what is the error in big-O notation?

Choice*

A) $O(x^4)$

B) $O(x^5)$

C) $O(x^3)$

D) $O((x-2)^3)$

E) $O((x-2)^4)$

Demo "Taylor of exp(x) about 2"

Making error predictions

Suppose you expand $\sqrt{x-10}$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.5$, you find that the Taylor truncation error is about 10^{-4}.

What is the Taylor truncation error for $h_2 = 0.25$?

$\text{Error}(h) = O(h^{n+1})$, where $n = 3$, i.e.

$$\text{Error}(h_1) \approx C \cdot h_1^4$$

$$\text{Error}(h_2) \approx C \cdot h_2^4$$

While not knowing C or lower order terms, we can use the ratio of h_2/h_1

$$\text{Error}(h_2) \approx C \cdot h_2^4 = C \cdot h_1^4 \left(\frac{h_2}{h_1}\right)^4 \approx \text{Error}(h_1) \cdot \left(\frac{h_2}{h_1}\right)^4$$

Can make prediction of the error for one h if we know another.

Using Taylor approximations to obtain derivatives

Let's say a function has the following Taylor series expansion about $x = 2$.

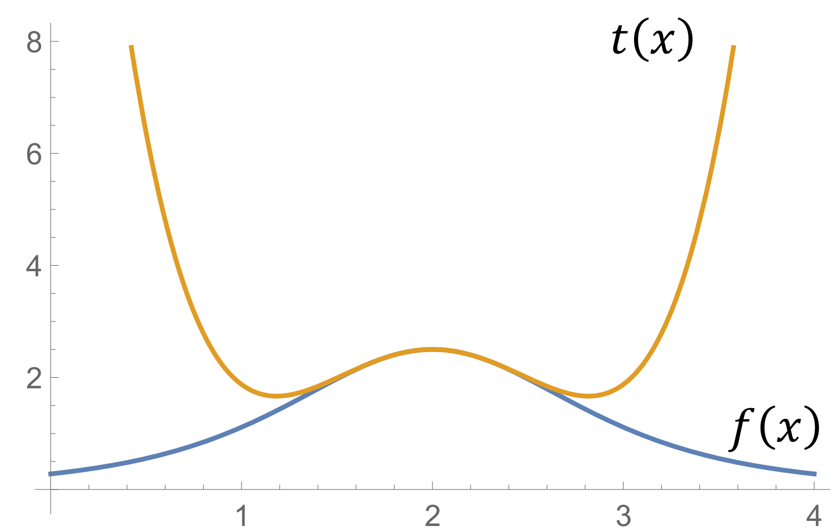
$$f(x) = \frac{5}{2} - \frac{5}{2}(x-2)^2 + \frac{15}{8}(x-2)^4 - \frac{5}{4}(x-2)^6 + \frac{25}{32}(x-2)^8 + O((x-2)^9)$$

Therefore the Taylor polynomial of order 4 is given by

$$t(x) = \frac{5}{2} - \frac{5}{2}(x-2)^2 + \frac{15}{8}(x-2)^4$$

where the first derivative is

$$t'(x) = -5(x-2) + \frac{15}{2}(x-2)^3$$



Using Taylor approximations to obtain derivatives

We can get the approximation for the derivative of the function $f(x)$ using the derivative of the Taylor approximation:

$$t'(x) = -5(x-2) + \frac{15}{2}(x-2)^3$$

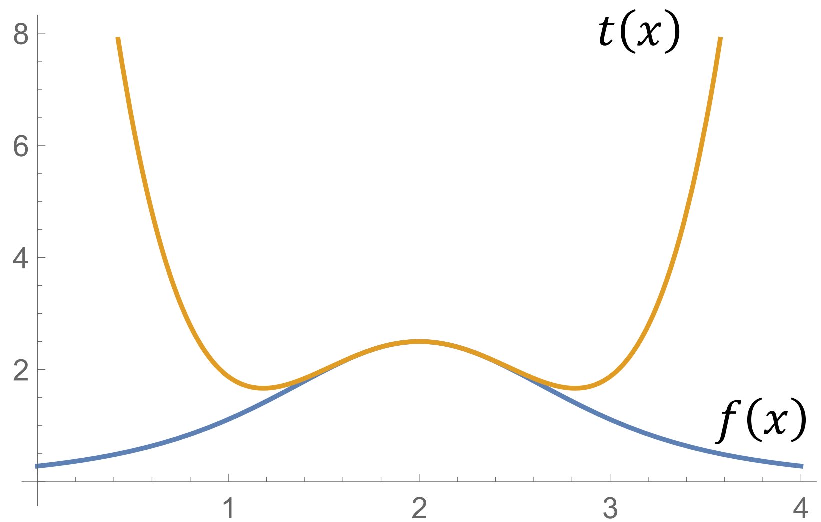
For example, the approximation for $f'(2.3)$ is

$$f'(2.3) \approx t'(2.3) = -1.2975$$

(note that the exact value is

$$f'(2.3) = -1.31444$$

What happens if we want to use the same method to approximate $f'(3)$?



Iclicker question

The function

$$f(x) = \cos(x)\, x^2 + \frac{\sin(2x)}{(x+2x^2)^3}$$

is approximated by the following Taylor polynomial of degree $n = 2$ about $x = 2\pi$

$$t_2(x) = 39.4784 \; + \; 12.5664\,(x - 2\pi) - \; 18.73922\,(x - 2\pi)^2$$

Determine an approximation for the first derivative of $f(x)$ at $x = 6.1$

A) 18.7741
B) 12.6856
C) 19.4319
D) 15.6840

Computing integrals using Taylor Series

A function $f(x)$ is approximated by a Taylor polynomial of order n around $x = 0$.

$$t_n = \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!}(x)^i$$

We can find an approximation for the integral $\int_s^t f(x)dx$ by integrating the polynomial:

$$\begin{aligned}\int_s^t f(x)dx &\approx \int_s^t a_0 + a_1 x + a_2 x^2 + a_3 x^3 dx \\ &= a_0 \int_s^t 1dx + a_1 \int_s^t x \cdot dx + a_2 \int_s^t x^2 dx + a_3 \int_s^t x^3 dx\end{aligned}$$

Where we can use $\int_s^t x^i\, dx = \frac{t^{i+1}}{i+1} - \frac{s^{i+1}}{i+1}$

Demo "Computing PI with Taylor"

Iclicker question

A function $f(x)$ is approximated by the following Taylor polynomial:

$$t_5(x) = 10 + x - 5\,x^2 - \frac{x^3}{2} + \frac{5x^4}{12} + \frac{x^5}{24} - \frac{x^6}{72}$$

Determine an approximated value for $\int_{-3}^{1} f(x)\,dx$

A) -10.27
B) -11.77
C) 11.77
D) 10.27

Finite difference approximation

For a given smooth function $f(x)$, we want to calculate the derivative $f'(x)$ at $x = 1$.

Suppose we don't know how to compute the analytical expression for $f'(x)$, but we have available a code that evaluates the function value:

```python
def f(x):
    # do stuff here
    feval = ...
    return feval
```

We know that:

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Can we just use $f'(x) \approx \frac{f(x+h)-f(x)}{h}$? How do we choose h? Can we get estimate the error of our approximation?

For a differentiable function $f\colon \mathcal{R} \to \mathcal{R}$, the derivative is defined as:

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

Let's consider the finite difference approximation to the first derivative as

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Where h is often called a "perturbation", i.e. a "small" change to the variable x. By the Taylor's theorem we can write:

$$f(x+h) = f(x) + f'(x)\,h + f''(\xi)\frac{h^2}{2}$$

For some $\xi \in [x, x+h]$. Rearranging the above we get:

$$f'(x) = \frac{f(x+h) - f(x)}{h} - f''(\xi)\frac{h}{2}$$

Therefore, the **truncation error** of the finite difference approximation is bounded by $\mathrm{M}\frac{h}{2}$, where M is a bound on $|f''(\xi)|$ for ξ near x.

Demo: Finite Difference

$f(x) = e^x - 2$

We want to obtain an approximation for $f'(1)$

$dfexact = e^x$

$$dfapprox = \frac{e^{x+h} - 2 - (e^x - 2)}{h}$$

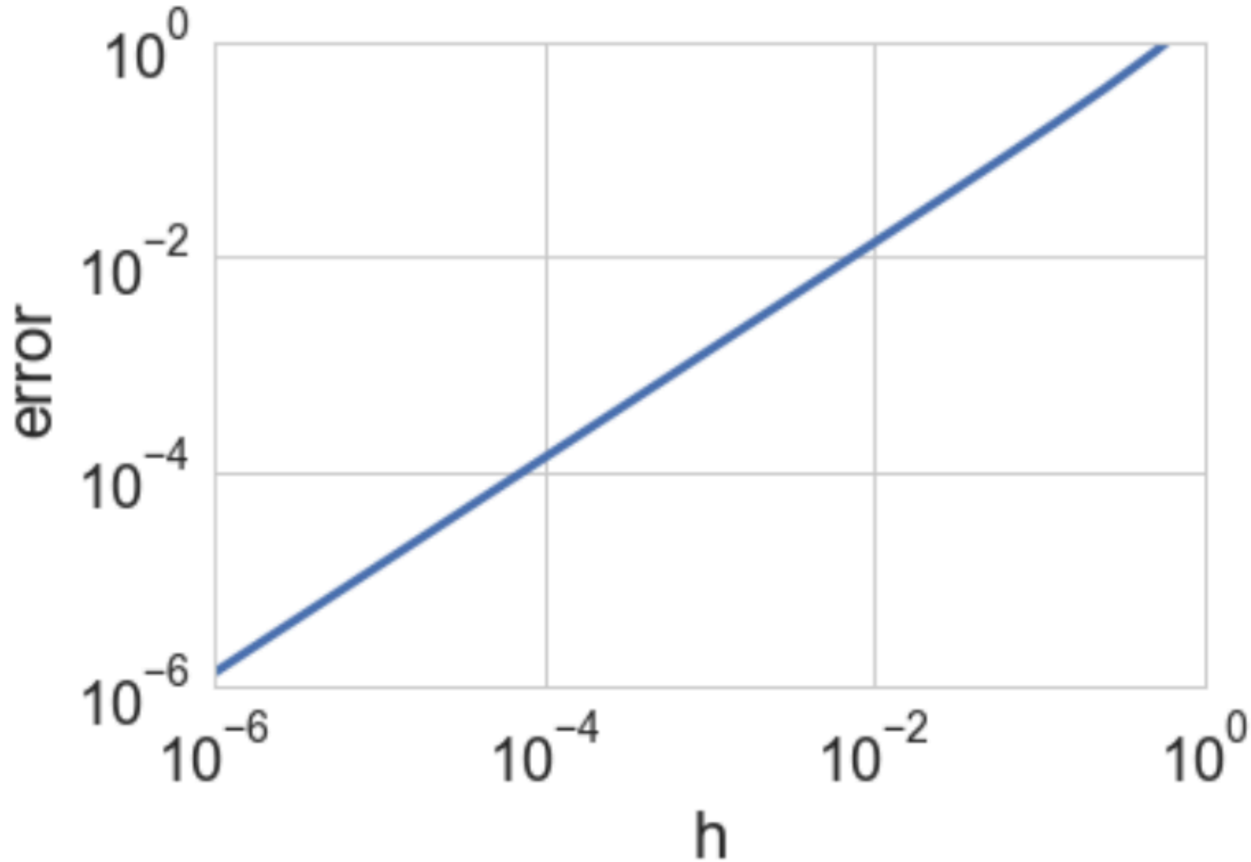
$error(h) = abs(dfexact - dfapprox)$

$$error < \left| f''(\xi)\frac{h}{2} \right|$$

truncation error

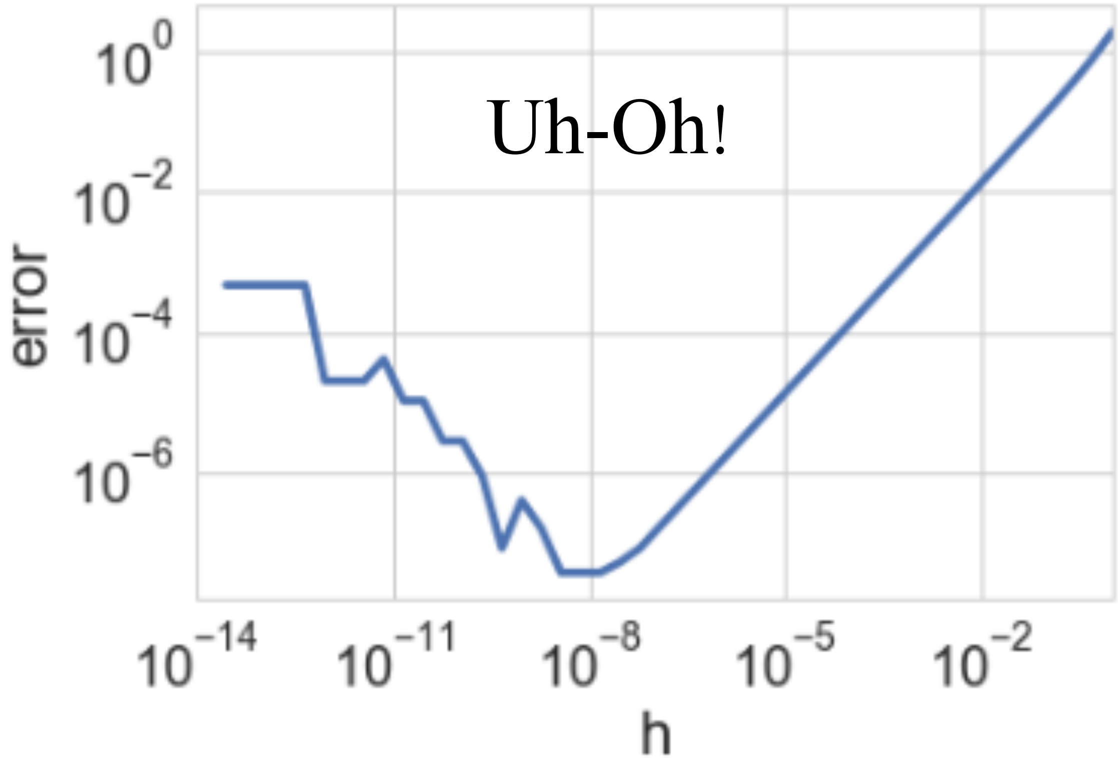
h	$error$
1.000000E+00	1.952492E+00
5.000000E-01	8.085327E-01
2.500000E-01	3.699627E-01
1.250000E-01	1.771983E-01
6.250000E-02	8.674402E-02
3.125000E-02	4.291906E-02
1.562500E-02	2.134762E-02
7.812500E-03	1.064599E-02
3.906250E-03	5.316064E-03
1.953125E-03	2.656301E-03
9.765625E-04	1.327718E-03
4.882812E-04	6.637511E-04
2.441406E-04	3.318485E-04
1.220703E-04	1.659175E-04
6.103516E-05	8.295707E-05
3.051758E-05	4.147811E-05
1.525879E-05	2.073897E-05
7.629395E-06	1.036945E-05
3.814697E-06	5.184779E-06
1.907349E-06	2.592443E-06

Demo: Finite Difference



$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Should we just keep decreasing the perturbation h, in order to approach the limit $h \to 0$ and obtain a better approximation for the derivative?



What happened here?

$$f(x) = e^x - 2$$

$$f'(x) = e^x \rightarrow f'(1) \approx 2.7$$

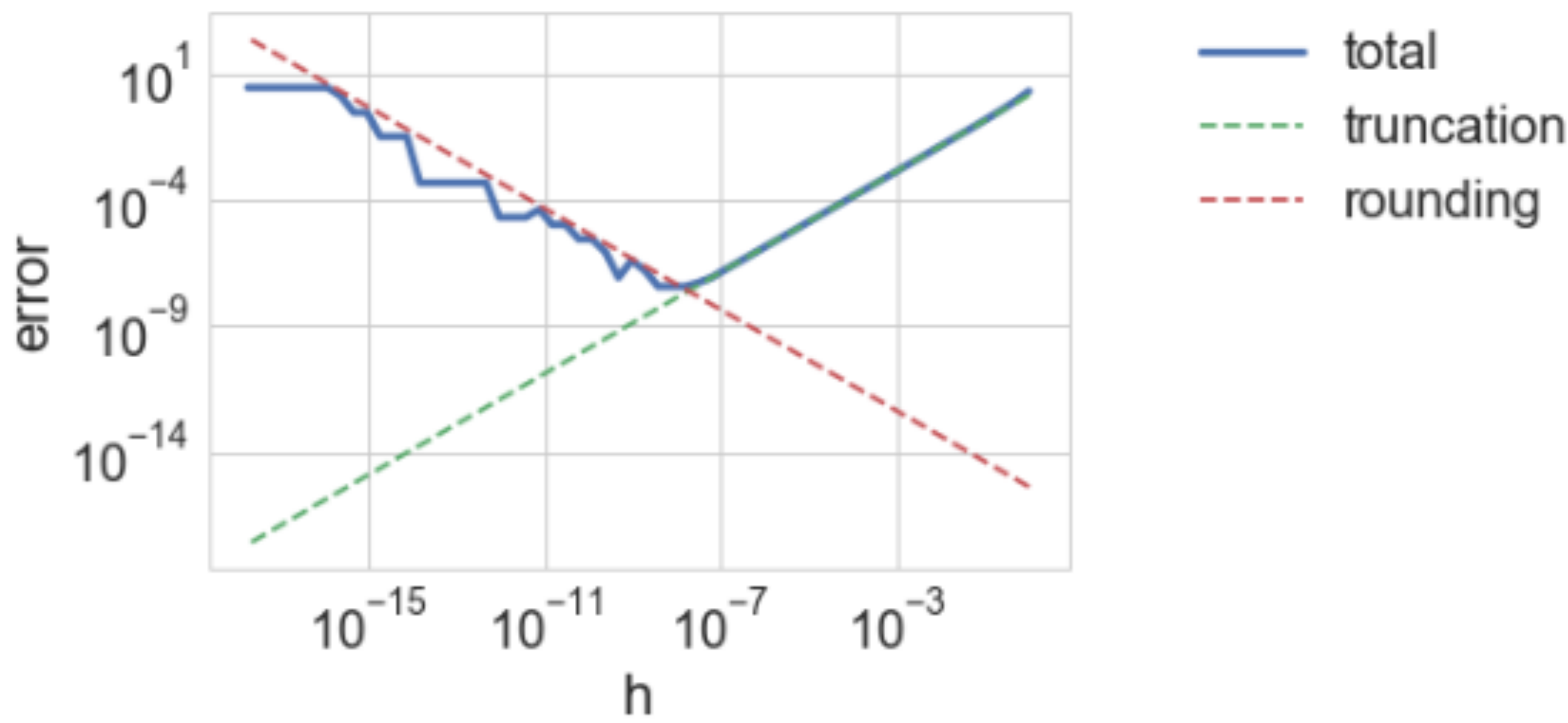
$$f'(1) = \lim_{h \to 0} \left(\frac{f(1+h) - f(1)}{h} \right)$$

Rounding error!

1) for a "very small" h $(h < \epsilon) \rightarrow f(1+h) = f(1) \rightarrow f'(1) = 0$

2) for other still "small" h $(h > \epsilon) \rightarrow f(1+h) - f(1)$ gives results with fewer significant digits

(We will later define the meaning of the quantity ϵ)



Truncation error: $error \sim M\frac{h}{2}$

Rounding error: $error \sim \frac{2\epsilon}{h}$

Minimize the error

$$\frac{2\epsilon}{h} + M\frac{h}{2}$$

Gives

$$h = 2\sqrt{\epsilon/M}$$