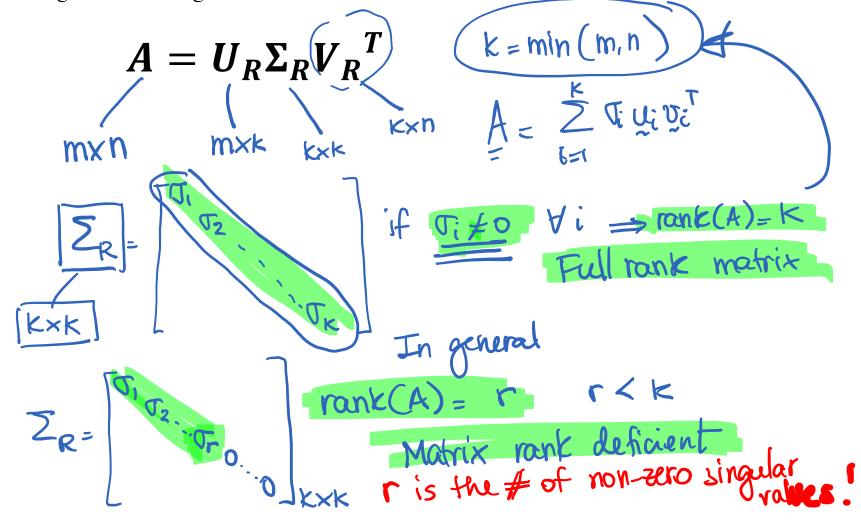
Singular Value Decomposition (applications)

1) Determining the rank of a matrix Suppose **A** is a $m \times n$ rectangular matrix where m > n: u_n σ_n u_1 nxn $egin{array}{cccc} & \sigma_1 \, \mathbf{v}_1^T & ... \ & & \vdots & \vdots \ & \sigma_n \, \mathbf{v}_n^T & ... \end{pmatrix}$ *A* = \boldsymbol{u}_1 Jn U General $A_{k} = \sum_{i=1}^{k} \nabla_{i} u_{i} \nabla_{i}$ $A_1 = \sigma_1 \mu_1 \nu_1$ rank(Ak) = K $A_2 = \nabla_2 U_2 \underbrace{v_2}^T + \nabla_1 U_1 \underbrace{v_1}^T$ **rank** (A₂) = 2 $rank(A_1) = 1$

Rank of a matrix

For general rectangular matrix A with dimensions $m \times n$, the reduced SVD is:



Rank of a matrix

- The rank of **A** equals the number of non-zero singular values which is the same as the number of non-zero diagonal elements in Σ .
- Rounding errors may lead to small but non-zero singular values in a rank deficient matrix, hence the rank of a matrix determined by the number of non-zero singular values is sometimes called "effective rank".
- The right-singular vectors (columns of V) corresponding to vanishing singular values span the null space of A.
- The left-singular vectors (columns of **U**) corresponding to the non-zero singular values of **A** span the range of **A**.

2) Pseudo-inverse A=UZVT Z=

Problem: if **A** is rank-deficient, Σ is not be invertible •

 $rank(A) = \Gamma$

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A = A

Σ^T-

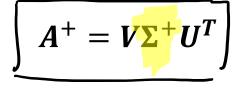
side note

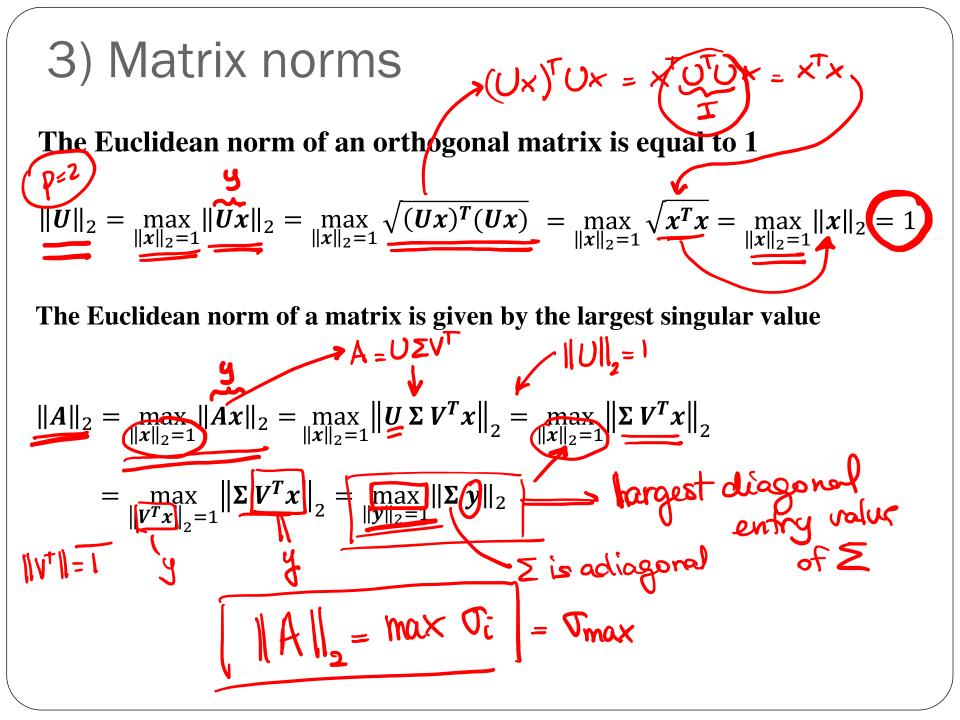
 $= V \Sigma^{-1} U$

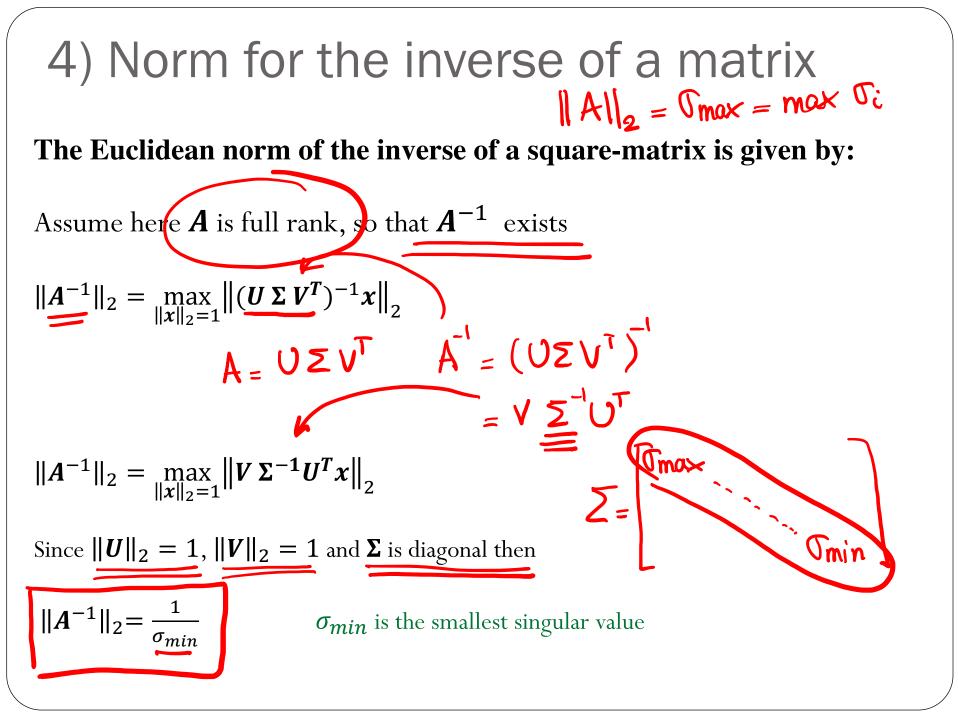
- How to fix it: Define the Pseudo Inverse
- **Pseudo-Inverse of a diagonal matrix**: •

$$(\mathbf{\Sigma}^+)_i = \begin{cases} \frac{1}{\sigma_i}, & \text{if } \sigma_i \neq 0\\ 0, & \text{if } \sigma_i = 0 \end{cases}$$

 $A = U \Sigma V^{T} (but A is)$ invertible $A' = (U \Sigma V^{T})' = (V^{T})' \Sigma' U^{T}$ **Pseudo-Inverse of a matrix** *A*:







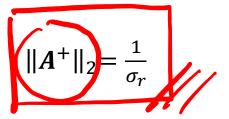
5) Norm of the pseudo-inverse matrix

 $A^+ = V\Sigma^+ U^T \checkmark$

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Smin

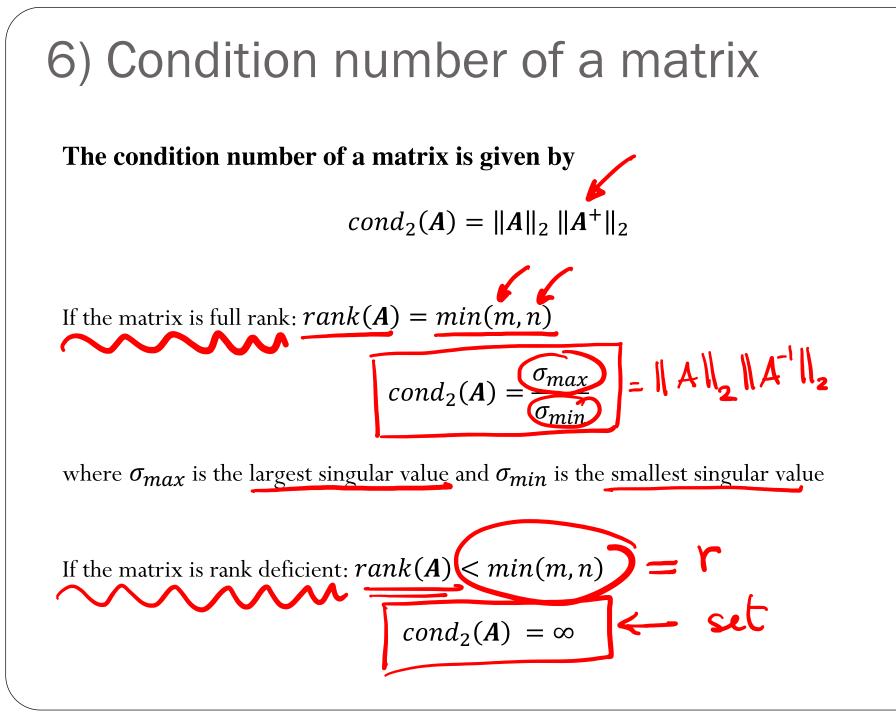
The norm of the pseudo-inverse of a $m \times n$ matrix is:



where σ_r is the smallest **non-zero** singular value. This is valid for any matrix, regardless of the shape or rank.

Note that for a full rank square matrix, $||A^+||_2$ is the same as $||A^{-1}||_2$.

Zero matrix: If **A** is a zero matrix, then A^+ is also the zero matrix, and $||A^+||_2 = 0$



7) Low-Rank Approximation

We will again use the SVD to write the matrix A as a sum of outer products (of left and right singular vectors) – here for m > n without loss of generality:

$$A = \begin{pmatrix} \vdots & \dots & \vdots \\ u_1 & \dots & u_m \\ \vdots & \dots & \vdots \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & \sigma_n \\ & & 0 \\ & \vdots \\ & & 0 \end{pmatrix} \begin{pmatrix} \dots & \mathbf{v}_1^T & \dots \\ \vdots & \vdots & \vdots \\ & & & \mathbf{v}_n^T & \dots \end{pmatrix}$$
$$= \begin{pmatrix} \vdots & \dots & \vdots \\ u_1 & \dots & u_n \\ \vdots & \dots & \vdots \end{pmatrix} \begin{pmatrix} \dots & \sigma_1 \mathbf{v}_1^T & \dots \\ \vdots & \vdots & \vdots \\ \dots & \sigma_n \mathbf{v}_n^T & \dots \end{pmatrix}$$
$$A = \sigma_1 u_1 \mathbf{v}_1^T + \sigma_2 u_2 \mathbf{v}_2^T + \dots + \sigma_n u_n \mathbf{v}_n^T$$
Full rank matrix

