Convergence plots and Big-O notation

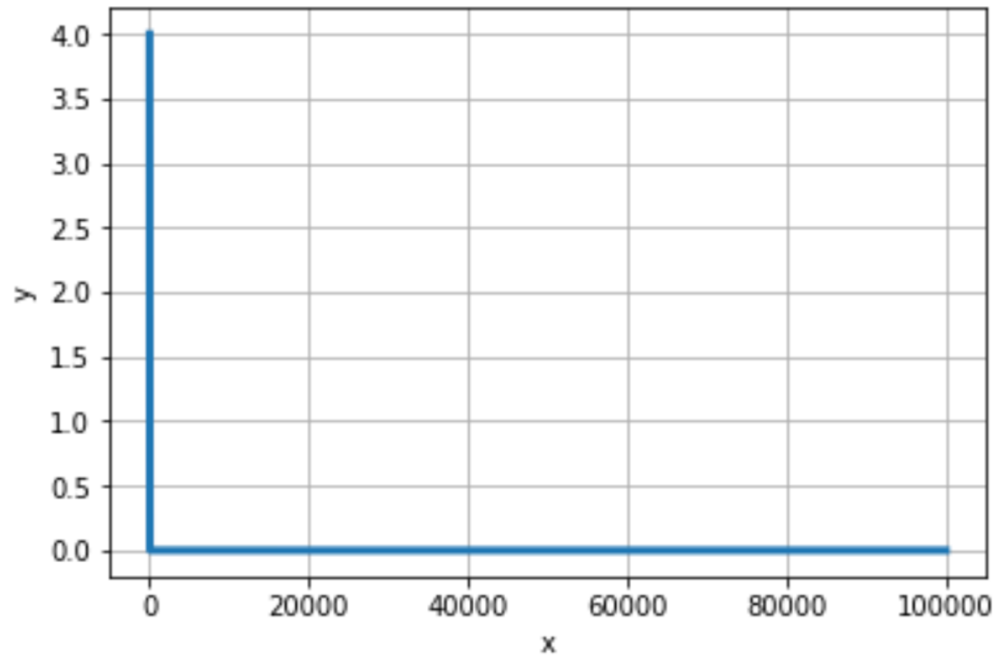
Let's first talk about plots…

- Power functions:

$$y = a\,x^b$$

$$\log y = \log\left(a\,x^b\right) = \log(a) + \log\left(\,x^b\right) = \log(a) + \mathrm{b}\log(x)$$

$$\bar{y} = \bar{a} + b\,\bar{x}$$



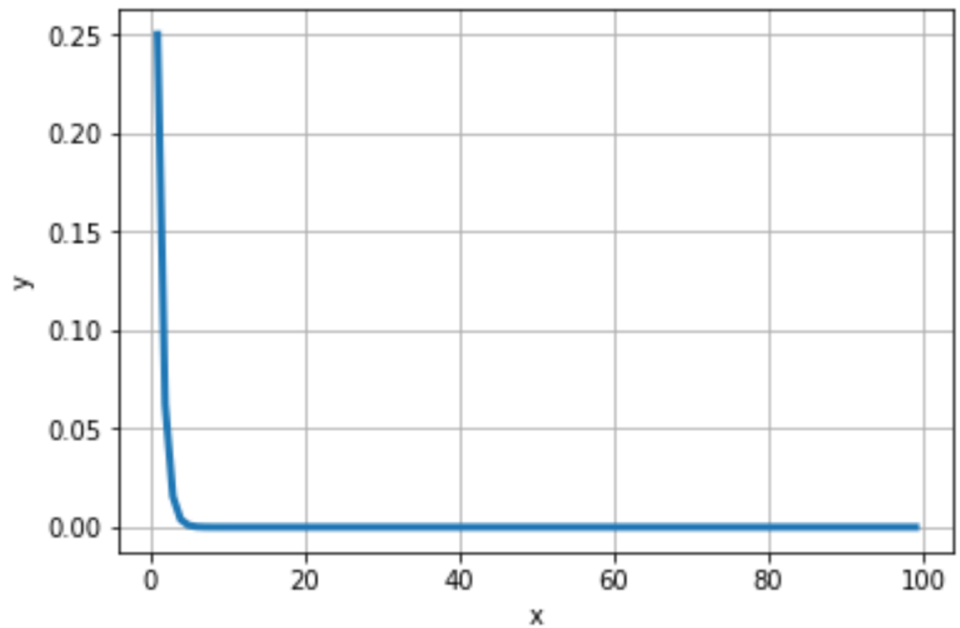
Let's first talk about plots…

- Exponential functions:

$$y = a\, b^x$$

$$\log y = \log(a\, b^x) = \log(a) + \log(\, b^x) = \log(a) + \ x \log(b)$$

$$\bar{y} = \bar{a} + \bar{b}\, x$$



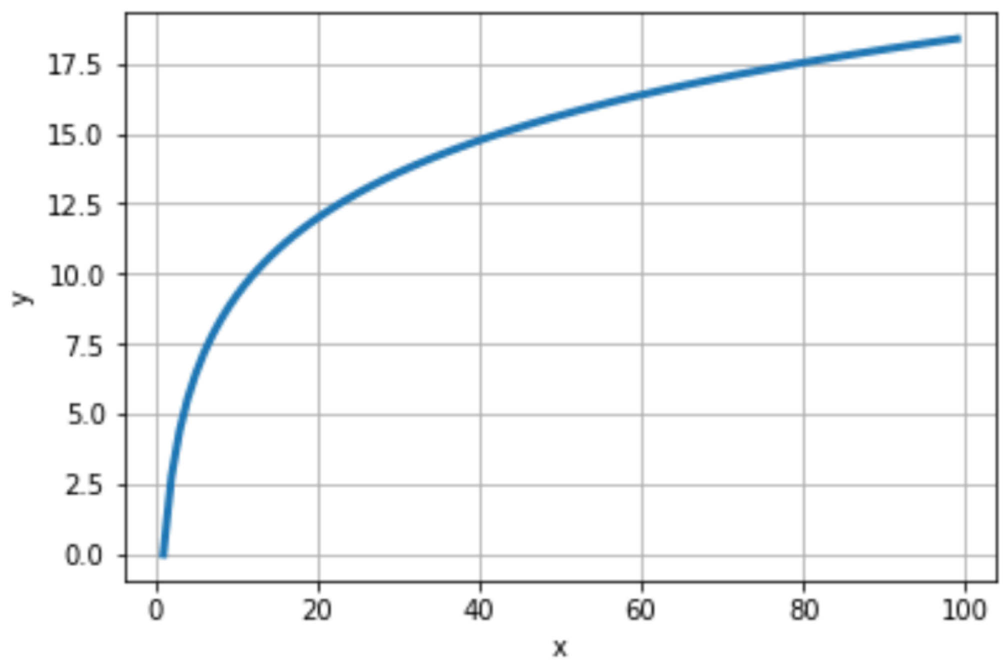
Let's first talk about plots…

- Log functions:

$$y = a\log(b\,x)$$

$$y = a\log(b) + a\log(x)$$

$$y = \bar{b} + a\,\bar{x}$$



Matrix-matrix multiplication example

For a matrix with dimensions $n \times n$, the computational complexity can be represented by a power function:

$$time = c\, n^a$$

We could count the total number of operations to determine the value of the constants above, but instead, we will get an estimate using a numerical experiment where we perform several matrix-matrix multiplications for vary matrix sizes, and store the time to take to perform the operation.

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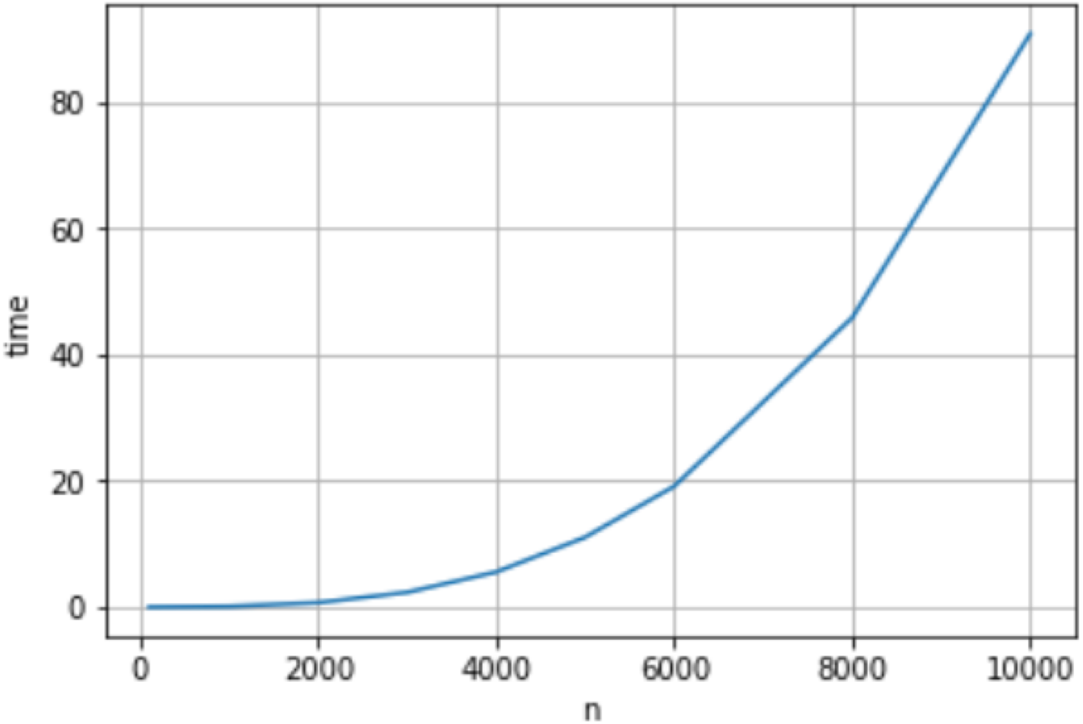
$$time = c\, n^a$$

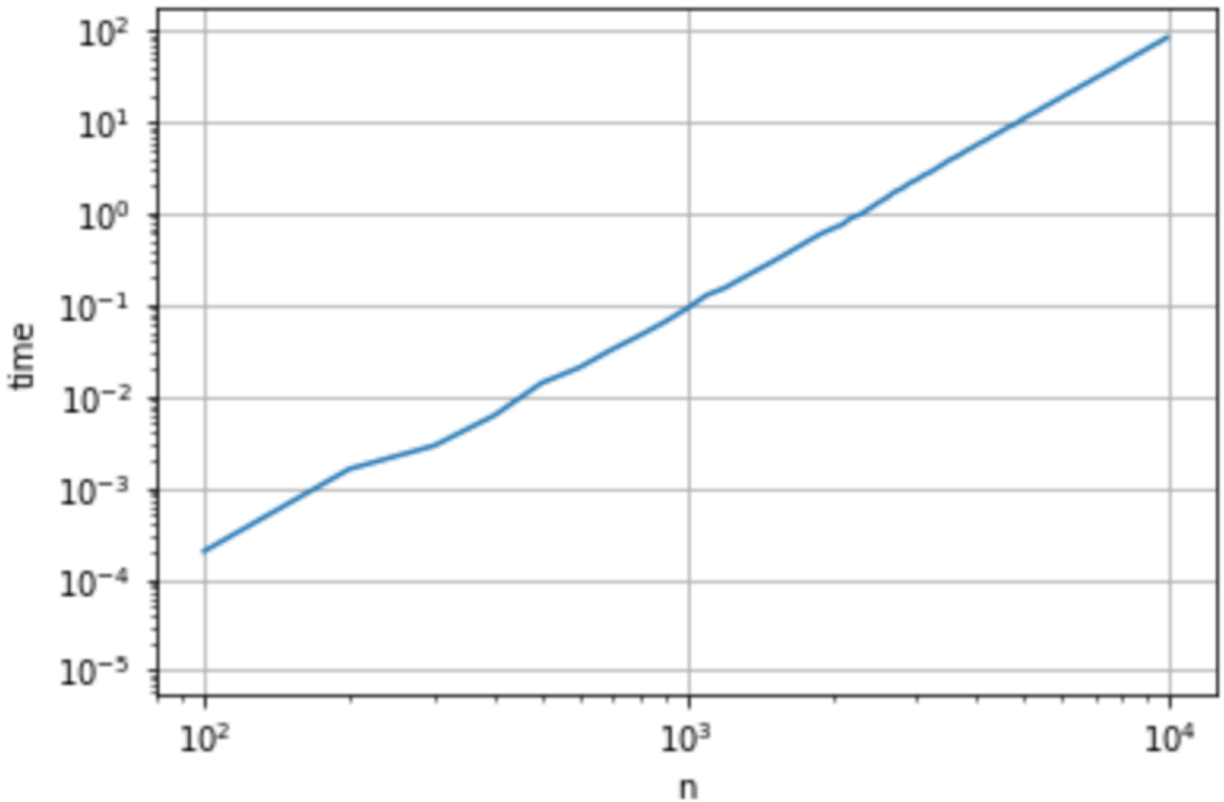


What type of plot will result in a straight line?

A) semilog-x B) semilog-y C)log-log

Power functions are represented by straight lines in a log-log plot, where the coefficient a is determined by the slope of the line.

$$time = c\, n^a$$



Demo: Cost of Matrix-Matrix Multiplication

Asymptotic Behavior; ("Big O") $O(\cdot)$ Notation

How do we say something exact without having to predict individual values exactly?

Let $g(n)$ be our model function.

Then instead of writing $\tau(n) \approx C \cdot g(n)$

We write $\tau(n) = O(g(n))$

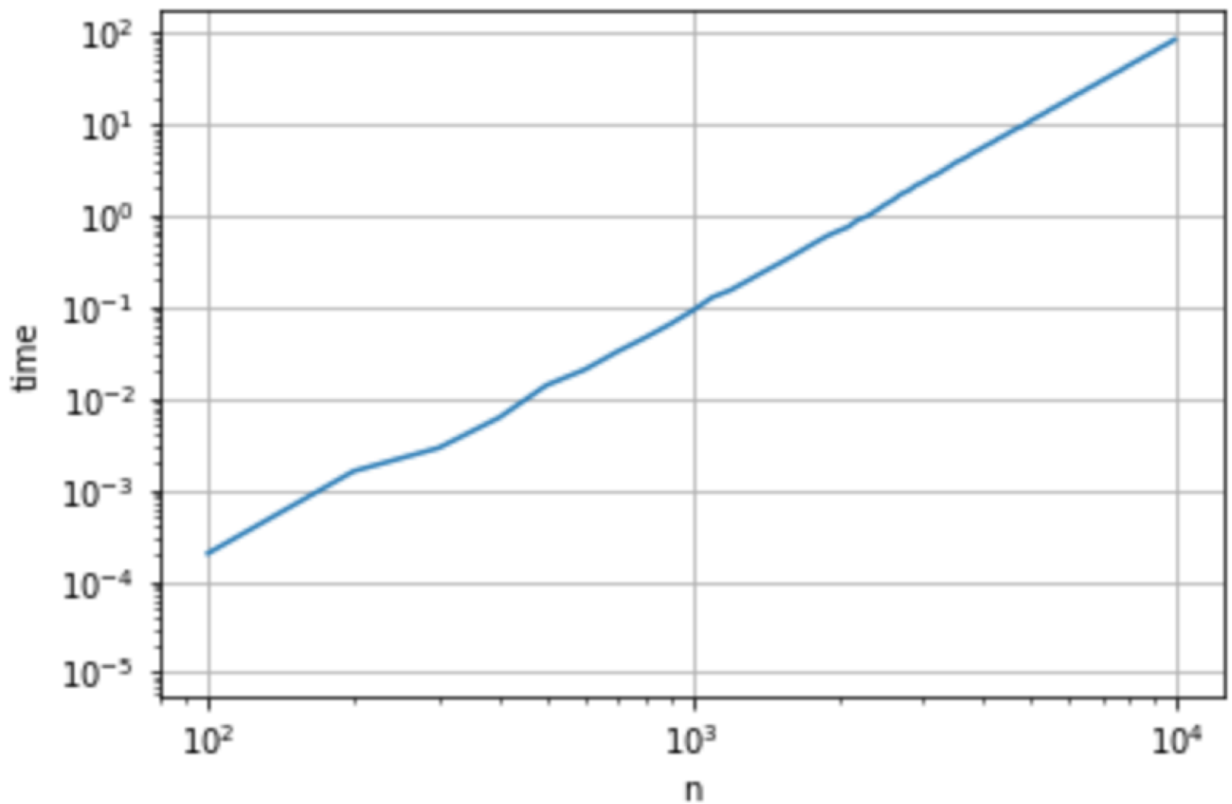
In other words, there is a constant C so that

$$\tau(n) \leq C \cdot g(n)$$

Instead of predicting time using $time = c\, n^a$, we can use the big-O notation to write

$$time = O(n^a)$$

where a can be obtained from the slope of the straight line.
For a matrix-matrix multiplication, what is the value of a?



Demo: Cost of Matrix-Matrix Multiplication

Revising Big-Oh notation

Let f and g be two functions. Then

$$f(x) = O\big(g(x)\big) \text{ as } x \to \infty$$

If an only if there is a positive constant M such that for all sufficiently large values of x, the absolute value of $f(x)$ is at most multiplied by the absolute value of $g(x)$. In other words, there exists a value M and some x_0 such that:

$$|f(x)| \leq M\,|g(x)| \quad \forall\, x \geq x_0$$

Example:

Consider the function $f(x) = 2x^2 + 27x + 1000$

When $x \to \infty$, the term x^2 is the most significant, and hence,

$$f(x) = O(x^2)$$

Revising Big-Oh notation

Let f and g be two functions. Then

$$f(x) = O\big(g(x)\big) \text{ as } x \to a$$

If an only if there exists a value M and some δ such that:

$$|f(x)| \leq M\ |g(x)| \quad \forall x \;\; where\ 0 < |x - a| < \delta$$

Same example…

Consider the function $f(x) = 2x^2 + 27x + 1000$

When $x \to 0$, the constant 1000 is the dominant part of the function. Hence,

$$f(x) = O(1)$$

Iclicker question

Suppose that the truncation error of a numerical method is given by the following function:

$$E(h) = 5h^2 + 3h$$

Which of the following functions are Oh-estimates of $E(h)$ as $h \to 0$

1) $O(5h^2)$
2) $O(h)$
3) $O(5h^2 + 3h)$
4) $O(h^2)$

Mark the correct answer:

A) 1 and 2

B) 2 and 3

C) 2 and 4

D) 3 and 4

E) NOTA

Iclicker question

Suppose that the complexity of a numerical method is given by the following function:

$$c(n) = 5n^2 + 3n$$

Which of the following functions are Oh-estimates of $c(n)$ as $n \to \infty$

1) $O(5n^2 + 3n)$
2) $O(n^2)$
3) $O(n^3)$
4) $O(n)$

Mark the correct answer:

A) 1,2,3

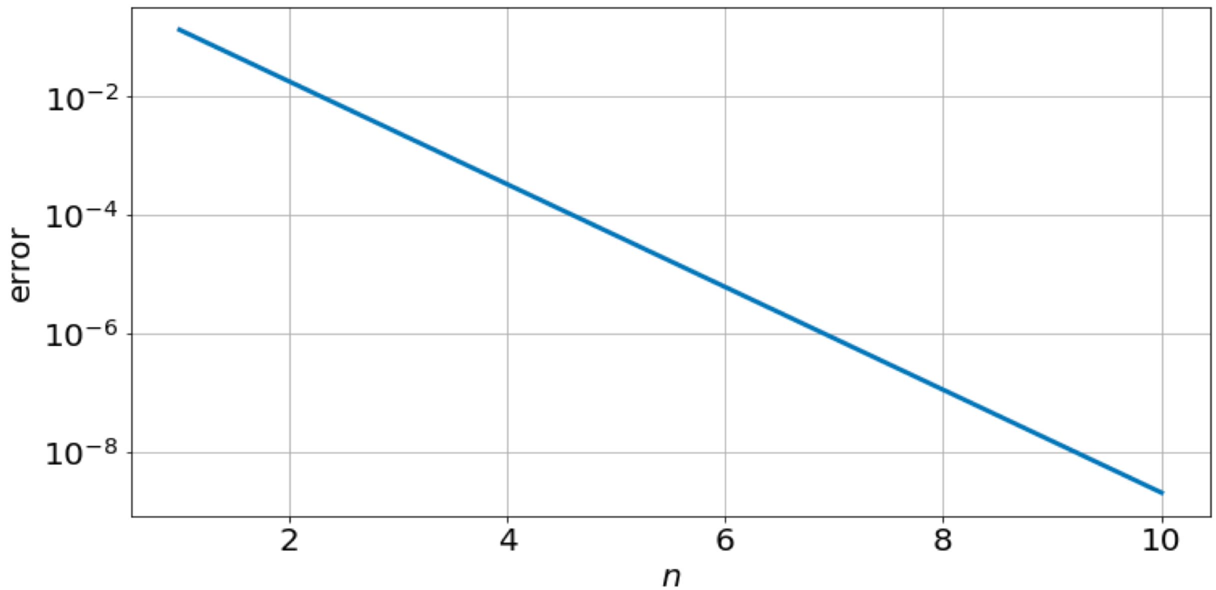
B) 1,2,3,4

C) 4

D) 3

E) NOTA

Select the function that best represents the decay of the error as n increases

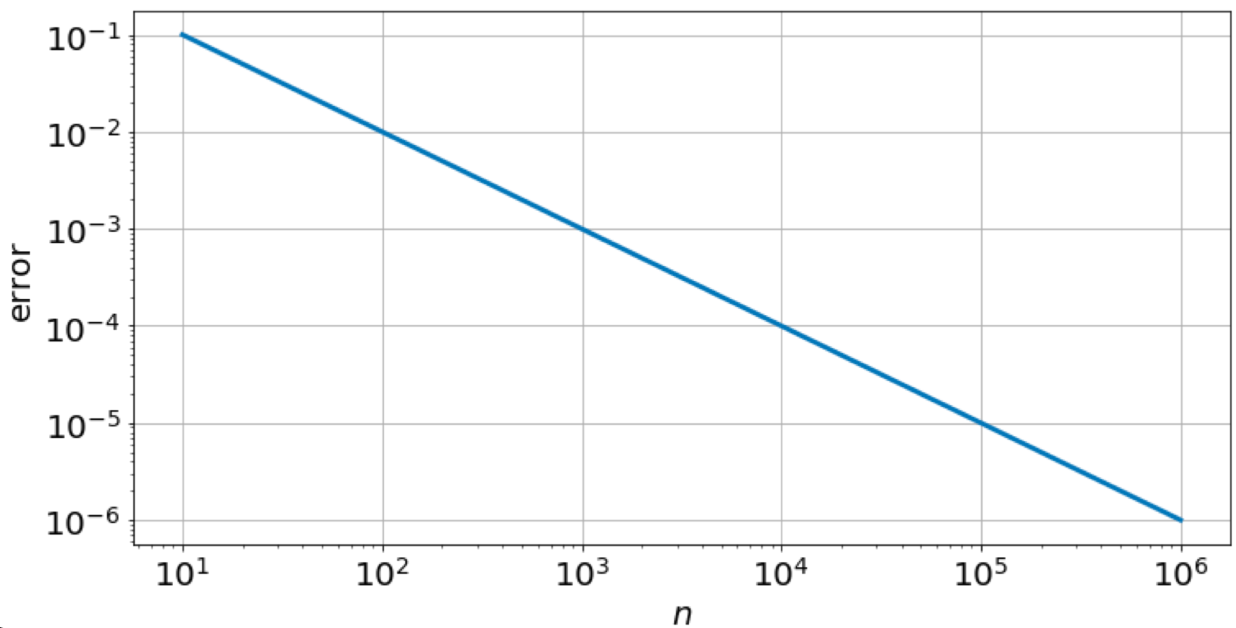


A) e^{-2n}

B) e^{-n}

C) n^{-1}

D) n^{-2}



A) e^{-2n}

B) e^{-n}

C) n^{-1}

D) n^{-2}

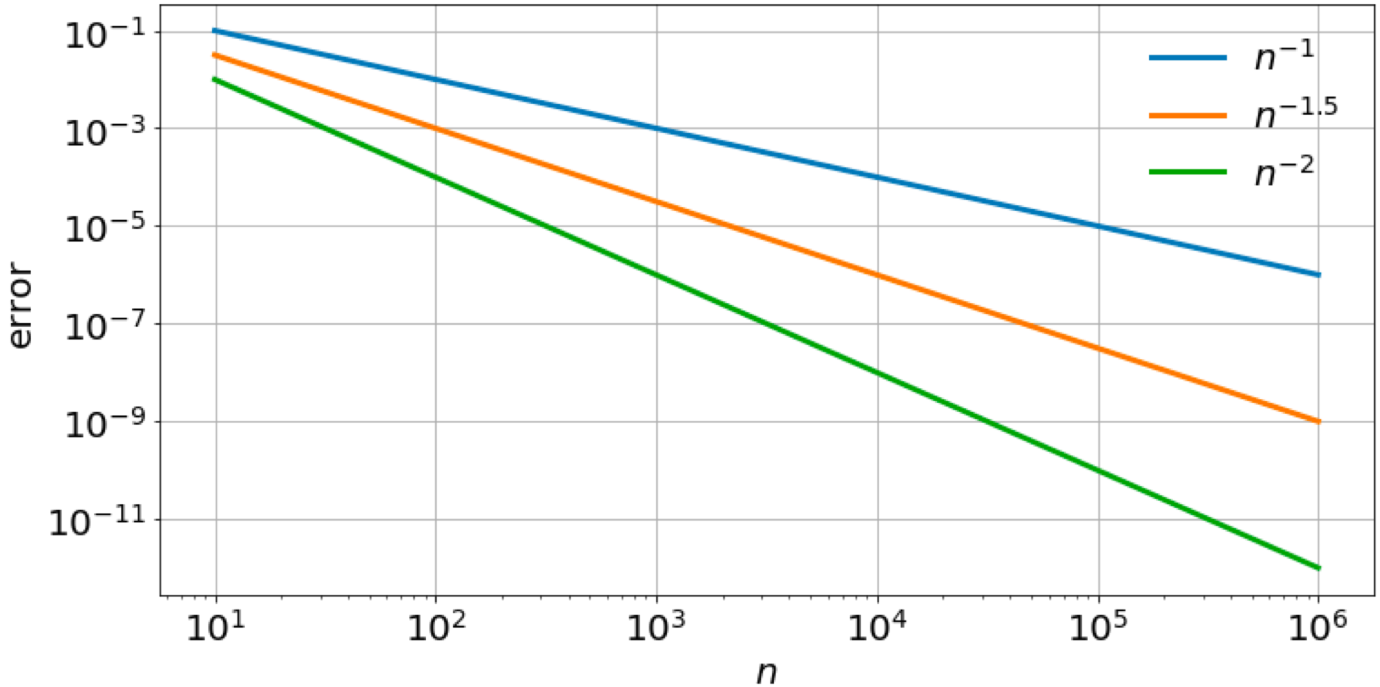
Rates of convergence

1) Algebraic convergence: $error \sim \frac{1}{n^\alpha}$ or $O\left(\frac{1}{n^\alpha}\right)$

Algebraic growth: $time \sim n^\alpha$ or $O(n^\alpha)$

α: Algebraic index of convergence

A sequence that grows or converges algebraically is a straight line in a log-log plot.



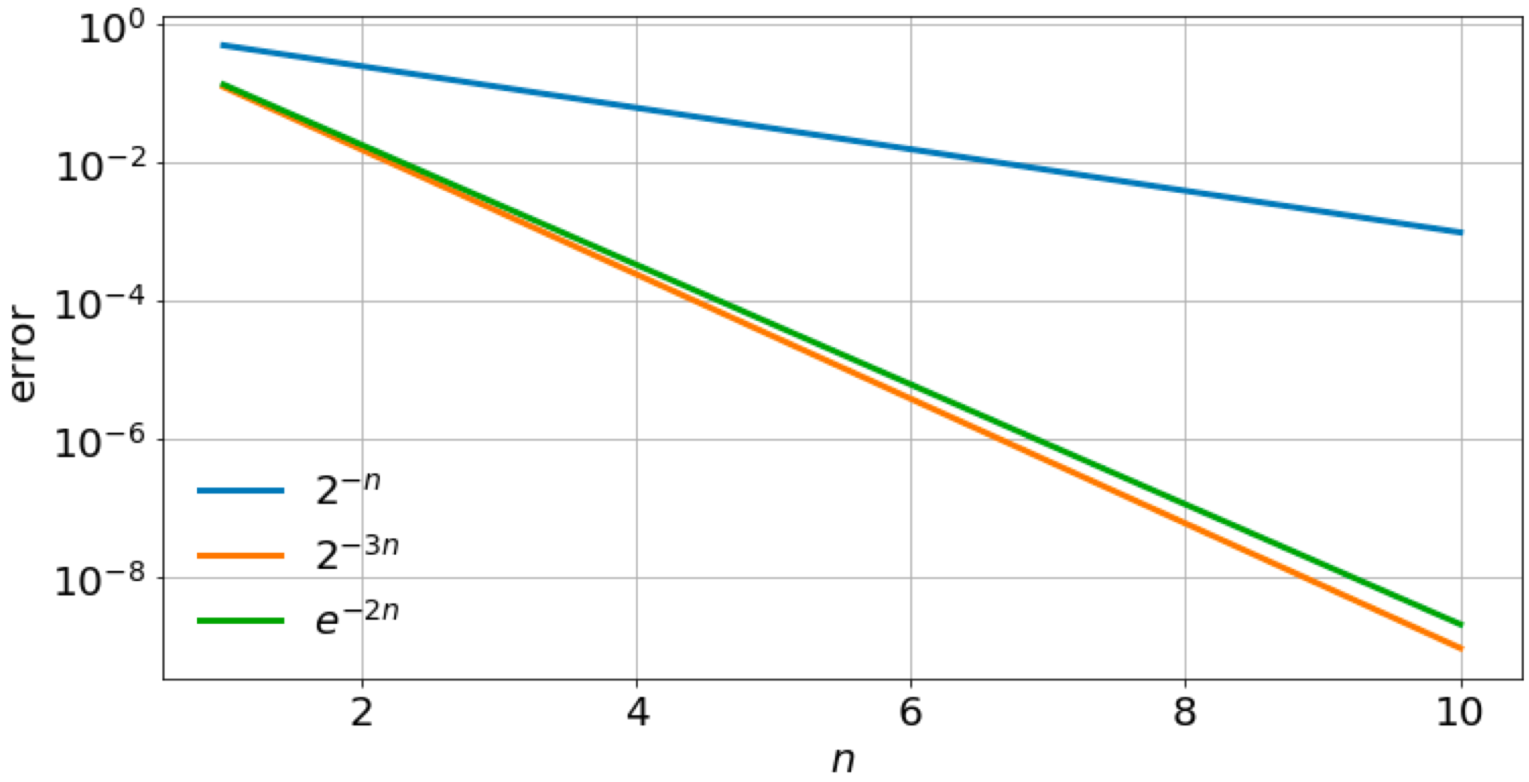
Demo "Exponential, Algebraic and Geometric convergence"

Rates of convergence

2) Exponential convergence: $error \sim e^{-\alpha n}$ or $O(e^{-\alpha n})$

Exponential growth: $time \sim e^{\alpha n}$ or $O(e^{\alpha n})$

A sequence that grows or converges exponentially is a straight line in a linear-log plot.



Rates of convergence

Exponential growth/convergence is much faster than algebraic growth/convergence.