1 What is Gradient and Poisson Equation

The gradient ∇f of a function f is a more general way to describe 'slope'. For example, in one-dimensional coordinates, $\nabla f = \frac{df}{dx}$ which is the slope of function f with respect to variable x. In three-dimensional coordinates (x, y, z) the derivative of f can be written as $\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$. Now given two functions ϕ and f, poisson equation is represented as

$$\nabla^2 \phi = f$$

Similarly, in three-dimensional coordinates, it can be written as $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = f$. We are basically saying that the second derivatives of a function ϕ is equal to f. In this lab, f is given and we are trying to find ϕ so that Poisson equation 1 is satisfied.

2 Why do we care?

- Gradient based techniques are used for some really interesting problems in image processing. Digital compositing or superimposing of images for example is used to place one photo into another such that they blend seamlessly. Take a look at figure 1. We have two images which have different image characteristics like color, texture, etc. When the two are simply placed on top of each other, the resulting image looks like the one on left in figure 2. However using our Poisson gradient technique, the eye image can be made to seamlessly blend into the hand image. Pretty cool, huh?

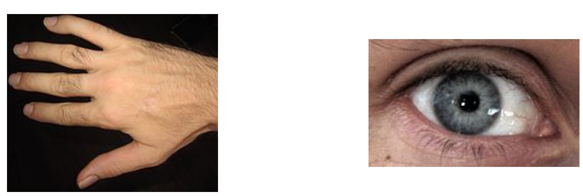

Figure 1: Hand and eye photos that need to be superimposed

- In the context of systems programming, solving the Poisson equation for image editing presents many opportunities to parallelize your implementation by launching multiple threads to solve the gradient function for different parts of your image. And that's exactly what we

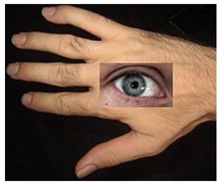 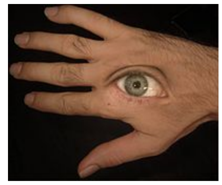

Figure 2: Traditional image copy pasting on left, using Poisson's equation techniques on right

are going to do here! :)

3 How to solve Poisson's equations? Lax-Friedrichs method

The Lax-Friedrichs method is a kind of numerical method we can use to solve Poisson equations. The basic idea of numerical solvers, which implement a numerical method, is to start by picking an initial point (a pair or tuple of values). We repeatedly update the values in a target matrix such that the error, i.e. the distance between current values and target values that solve equation 1 above, decreases. By repeating the method until this error is acceptably small we get a result that is very close to the exact answer. For example, in three-dimensional coordinates (x, y, z), an initial point is a tuple (x_0, y_0, z_0). Suppose the target values that solve the equation are (x_T, y_T, z_T), then initial error $\epsilon_0 = distance((x_0, y_0, z_0), (x_T, y_T, z_T))$. By using the method implemented in the solver, we get a new tuple (x_1, y_1, z_1). We then recalculate the error and update current values. Repeat until the error $\epsilon_t < Threshold$ or some threshold number of iterations is reached.