Question

- Which of the following are represented by the hexadecimal number 0x00494824 ?

  – the integer 4802596
  – the string "$HI"
  – the float 6.7298704e-39
  – the instruction and $9, $2, $9

Pick up Handout!

Answer

- Which of the following are represented by the hexadecimal number 0x00494824 ?

  Answer: All of them. (See data.s) They are just different interpretations of the same bit patterns.

  (note: the string representation depends on endianness)

- Then how does the machine know which interpretation you want?

Answer

- Which of the following are represented by the hexadecimal number 0x00494824 ?

Answer: All of them. (See data.s) They are just different interpretations of the same bit patterns.

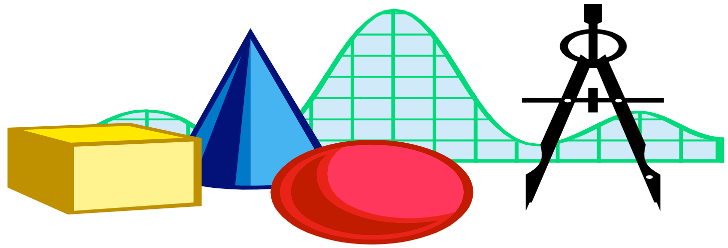
(note: the string representation depends on endianness)

- Then how does the machine know which interpretation you want?

You have to explicitly tell the machine which interpretation you want.

- Use an **integer load** (lw) to interpret them as an **int**
- Use a **floating point load** (l.s) to interpret them as a **float**
- Use a **branch** or a **jump**(bne or j) to interpret them as an **instruction**

Floating-point arithmetic



- Two's complement and floating point are the two standard number representations.
  - Floating point greatly simplifies working with large (e.g., 2^{70}) and small (e.g., 2^{-17}) numbers
  - Early machines did it in software with "scaling factors"
- We'll focus on the IEEE 754 standard for floating-point arithmetic.
  - How FP numbers are represented
  - Limitations of FP numbers
  - FP addition and multiplication

Floating-point representation

- IEEE numbers are stored using a kind of scientific notation.

$$\pm \text{mantissa} * 2^{\text{exponent}}$$

7.3×10^4

- We can represent floating-point numbers with three binary fields: a sign bit s, an exponent field e, and a fraction field f.

s	e	f

- The IEEE 754 standard defines several different precisions.
  - Single precision numbers include an 8-bit exponent field and a 23-bit fraction, for a total of 32 bits.
  - Double precision numbers have an 11-bit exponent field and a 52-bit fraction, for a total of 64 bits.

Sign

s	e	f

- The sign bit is 0 for positive numbers and 1 for negative numbers.
- But unlike integers, IEEE values are stored in signed magnitude format.

Mantissa

s	e	f

- The field f contains a binary fraction.
- The actual mantissa of the floating-point value is (1 + f).
  - In other words, there is an implicit 1 to the left of the binary point.
  - For example, if f is 01101…, the mantissa would be 1.01101…
- There are many ways to write a number in scientific notation, but there is always a *unique* normalized representation, with exactly one non-zero digit to the left of the point.

$$0.232 * 10^3 = 23.2 * 10^1 = 2.32 * 10^2 = \ldots$$

- A side effect is that we get a little more precision: there are 24 bits in the mantissa, but we only need to store 23 of them.

Exponent

s	e	f

- The e field represents the exponent as a biased number.
  - It contains the actual exponent *plus* 127 for single precision, or the actual exponent *plus* 1023 in double precision.
  - This would convert all single-precision exponents from –127 to +128 into unsigned numbers from 0 to 255, and all double-precision exponents from –1023 to +1024 into unsigned numbers from 0 to 2047.
- Two examples with single-precision numbers are shown below.
  - If the exponent is 4, the e field will be 4 + 127 = 131 (10000011_2).
  - If e contains 01011101 (93_{10}), the actual exponent is 93 – 127 = –34.
- Storing a biased exponent *before* a normalized mantissa means we can compare IEEE values as if they were signed integers.

Converting an IEEE 754 number to decimal

s	e	f

- The decimal value of an IEEE number is given by the formula:

$$(1 - 2s) * (1 + f) * 2^{e\text{-}bias}$$

- Here, the s, f and e fields are assumed to be in decimal.
  - (1 – 2s) is 1 or –1, depending on whether the sign bit is 0 or 1.
  - We add an implicit 1 to the fraction field f, as mentioned earlier.
  - Again, the bias is either 127 or 1023, for single or double precision.

Converting a decimal number to IEEE 754

- What is the single-precision representation of 347.625?

1. First convert the number to binary: 347.625 = 101011011.101_2.
2. Normalize the number by shifting the binary point until there is a single 1 to the left:

$$101011011.101 \times 2^0 =$$

3. The bits to the right of the binary point comprise the fractional field f.
4. The number of times you shifted gives the exponent. The field e should contain: exponent + 127.
5. Sign bit: 0 if positive, 1 if negative.

Example IEEE-decimal conversion

- Let's find the decimal value of the following IEEE number.

1 01111100 11000000000000000000000

Example IEEE-decimal conversion

- Let's find the decimal value of the following IEEE number.

  1 01111100 11000000000000000000000

- First convert each individual field to decimal.
  - The sign bit s is 1.
  - The e field contains 01111100 = 124_{10}.
  - The mantissa is 0.11000… = 0.75_{10}.
- Then just plug these decimal values of s, e and f into our formula.

$$(1 - 2s) * (1 + f) * 2^{e\text{-bias}}$$

- This gives us $(1 - 2) * (1 + 0.75) * 2^{124-127} = (-1.75 * 2^{-3}) = -0.21875$.

Special values

- The smallest and largest possible exponents e=00000000 and e=11111111 (and their double precision counterparts) are reserved for special values.
- If the mantissa is always (1 + f), then how is 0 represented?
  - The fraction field f should be 0000...0000.
  - The exponent field e contains the value 00000000.
  - With signed magnitude, there are *two* zeroes: +0.0 and –0.0.
- There are representations of positive and negative infinity, which might sometimes help with instances of overflow.
  - The fraction f is 0000...0000.
  - The exponent field e is set to 11111111.
- Finally, there is a special “not a number” value, which can handle some cases of errors or invalid operations such as 0.0/0.0.
  - The fraction field f is set to any non-zero value.
  - The exponent e will contain 11111111.

NaN

$\frac{0}{0}$

Range of single-precision numbers

What is the smallest positive single-precision value that can be represented?

$e = 00000001$

$f = 00000 \cdots 0$

Mantisa = 1.00000×2^{-126}

$e - 127 = 1 - 127 = -126$

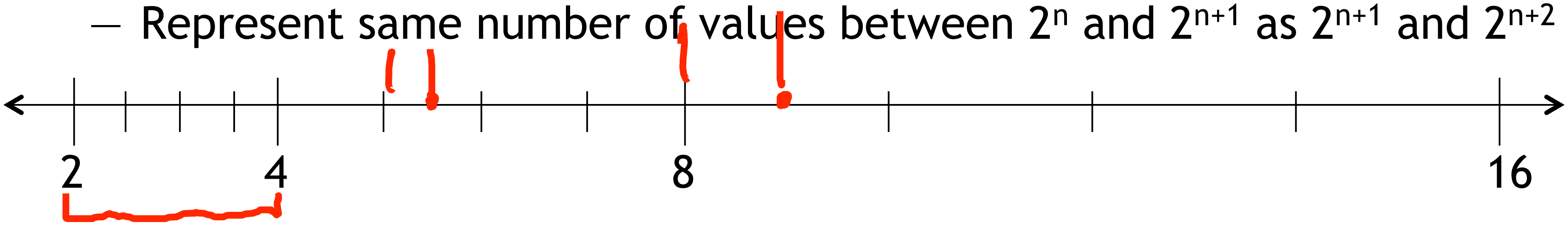
Range of single-precision numbers

$$(1 - 2s) * (1 + f) * 2^{e-127}.$$

- And the smallest *positive* non-zero number is $1 * 2^{-126} = 2^{-126}$.
  - The smallest e is 00000001 (1).
  - The smallest f is 00000000000000000000000 (0).
- The largest possible "normal" number is $(2 - 2^{-23}) * 2^{127} = 2^{128} - 2^{104}$.
  - The largest possible e is 11111110 (254).
  - The largest possible f is 11111111111111111111111 $(1 - 2^{-23})$.
- In comparison, the range of possible 32-bit integers in two's complement are only -2^{31} and $(2^{31} - 1)$
- How can we represent so many more values in the IEEE 754 format, even though we use the same number of bits as regular integers?

Finiteness

- There *aren't* more IEEE numbers.
- With 32 bits, there are $2^{32}-1$, or about 4 billion, different bit patterns.
  - These can represent 4 billion integers *or* 4 billion reals.
  - But there are an infinite number of reals, and the IEEE format can only represent *some* of the ones from about -2^{128} to $+2^{128}$.
  - Represent same number of values between 2^n and 2^{n+1} as 2^{n+1} and 2^{n+2}



- Thus, floating-point arithmetic has "issues"
  - Small roundoff errors can accumulate with multiplications or exponentiations, resulting in big errors.
  - Rounding errors can invalidate many basic arithmetic principles such as the associative law, $(x + y) + z = x + (y + z)$.
- The IEEE 754 standard guarantees that all machines will produce the same results—but those results may not be mathematically correct!

Limits of the IEEE representation

- Even some integers cannot be represented in the IEEE format.

```
int x   = 33554431;
float y = 33554431;
printf( "%d\n", x );
printf( "%f\n", y );

33554431
33554432.000000
```

- Some simple decimal numbers cannot be represented exactly in binary to begin with.

$$0.10_{10} = 0.0001100110011\ldots_2$$

0.10

- During the Gulf War in 1991, a U.S. Patriot missile failed to intercept an Iraqi Scud missile, and 28 Americans were killed.
- A later study determined that the problem was caused by the inaccuracy of the binary representation of 0.10.
  - The Patriot incremented a counter once every 0.10 seconds.
  - It multiplied the counter value by 0.10 to compute the actual time.
- However, the (24-bit) binary representation of 0.10 actually corresponds to 0.0999999046325683593750, which is off by 0.000000095367431640625.
- This doesn't seem like much, but after 100 hours the time ends up being off by 0.34 seconds—enough time for a Scud to travel 500 meters!
- UIUC Emeritus Professor Skeel wrote a short article about this.

[Roundoff Error and the Patriot Missile. SIAM News, 25(4):11, July 1992.]

Floating-point addition example

- To get a feel for floating-point operations, we'll do an addition example.
  - To keep it simple, we'll use base 10 scientific notation.
  - Assume the mantissa has four digits, and the exponent has one digit.
- The text shows an example for the addition:

$$99.99 + 0.161 = 100.151$$

- As normalized numbers, the operands would be written as:

$9.999 * 10^{1}$ $1.610 * 10^{-1}$

Steps 1-2: the actual addition

1. Equalize the exponents.

   The operand with the smaller exponent should be rewritten by increasing its exponent and shifting the point leftwards.

   $$1.610 * 10^{-1} =$$

   With four significant digits, this gets rounded to:

   This can result in a loss of least significant digits—the rightmost 1 in this case. But rewriting the number with the larger exponent could result in loss of the *most* significant digits, which is much worse.

2. Add the mantissas.

$$\begin{array}{rrcl} & 9.999 & * & 10^{1} \\ + & 0.016 & * & 10^{1} \\ \hline \end{array}$$

Steps 3-5: representing the result

3. Normalize the result if necessary.

$$10.015 * 10^1 =$$

   This step may cause the point to shift either left or right, and the exponent to either increase or decrease.

4. Round the number if needed.

$$1.0015 * 10^2 \text{ gets rounded to}$$

5. Repeat Step 3 if the result is no longer normalized.

   We don't need this in our example, but it's possible for rounding to add digits—for example, rounding 9.9995 yields 10.000.

Our result is , or . The correct answer is 100.151, so we have the right answer to four significant digits, but there's a small error already.

Multiplication

- To multiply two floating-point values, first multiply their magnitudes and add their exponents.

$$
\begin{array}{rrcl}
 & 9.999 & * & 10^{1} \\
* & 1.610 & * & 10^{-1} \\
\hline
 & 16.098 & * & 10^{0}
\end{array}
$$

- You can then round and normalize the result, yielding $1.610 * 10^{1}$.
- The sign of the product is the exclusive-or of the signs of the operands.
  - If two numbers have the same sign, their product is positive.
  - If two numbers have different signs, the product is negative.

$0 \oplus 0 = 0$ $\quad 0 \oplus 1 = 1$ $\quad 1 \oplus 0 = 1$ $\quad 1 \oplus 1 = 0$

- This is one of the main advantages of using signed magnitude.

Feedback

- Write one or more of the following: (name optional)

1. What you like about this class so far.

2. What we can do to make this class a better learning environment for you.

3. Something that we are doing that is detrimental to your learning and should stop.

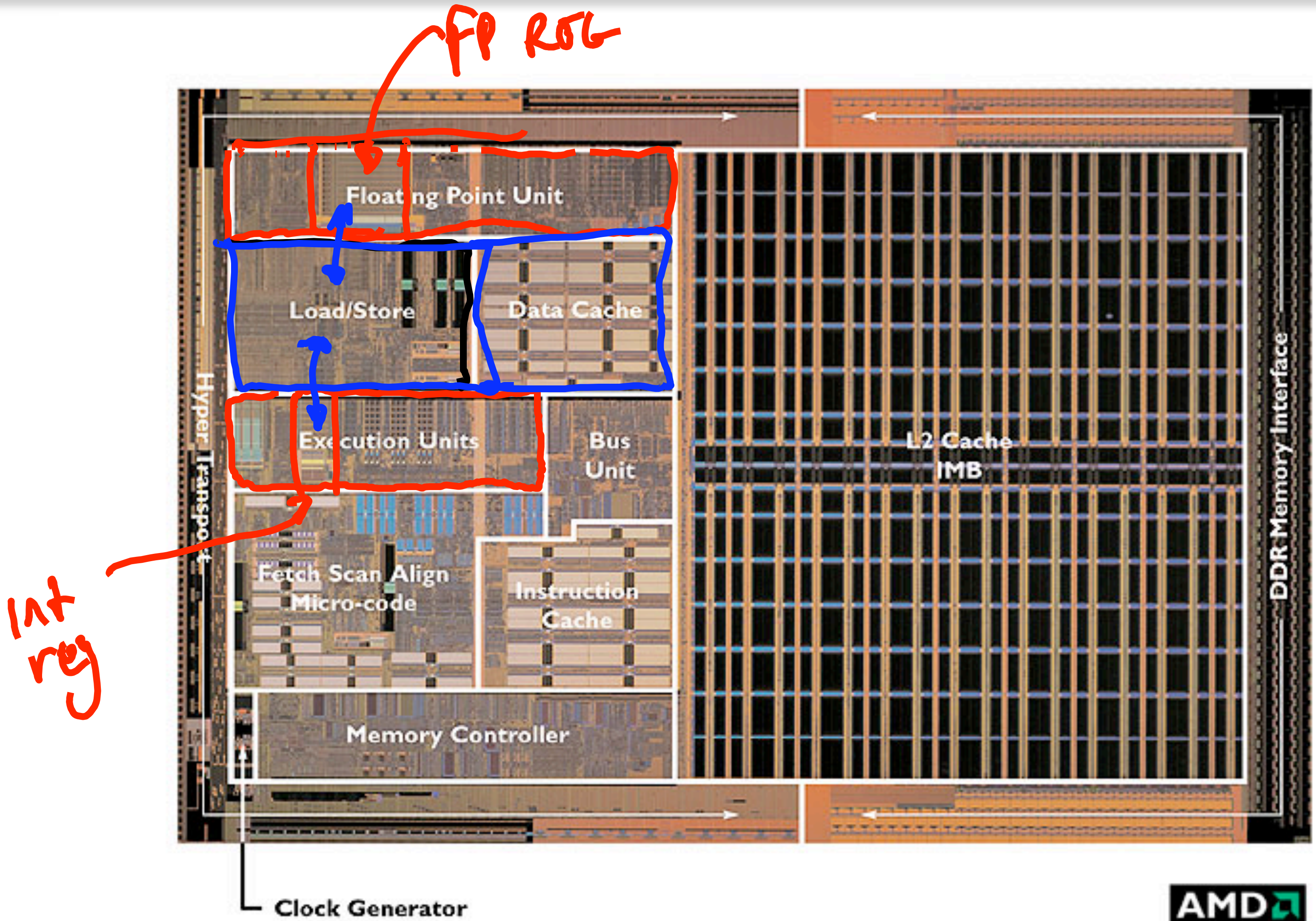
The history of floating-point computation

- In the past, each machine had its own implementation of floating-point arithmetic hardware and/or software.
  - It was impossible to write portable programs that would produce the same results on different systems.
- It wasn't until 1985 that the IEEE 754 standard was adopted.
  - Having a standard at least ensures that all compliant machines will produce the same outputs for the same program.

Floating-point hardware

- When floating point was introduced in microprocessors, there wasn't enough transistors on chip to implement it.
  - You had to buy a floating point co-processor (e.g., the Intel 8087)
- As a result, many ISA's use separate registers for floating point.
- Modern transistor budgets enable floating point to be on chip.
  - Intel's 486 was the first x86 with built-in floating point (1989)
- Even the newest ISA's have separate register files for floating point.
  - Makes sense from a floor-planning perspective.

FPU like co-processor on chip



Summary

- The IEEE 754 standard defines number representations and operations for floating-point arithmetic.
- Having a finite number of bits means we can't represent all possible real numbers, and errors will occur from approximations.