Data Structures and Algorithms
Hashing 2

CS 225
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Learning Objectives

Review fundamentals of hash tables

Introduce closed hashing approaches to hash collisions

Determine when and how to resize a hash table

Justify when to use different index approaches

wednesday

A Hash Table based Dictionary

User Code (is a map):

```
1  Dictionary<KeyType, ValueType> d;
2  d[k] = v;
```

A **Hash Table** consists of three things:

1. A hash function — Key → Int
   → must be hashable

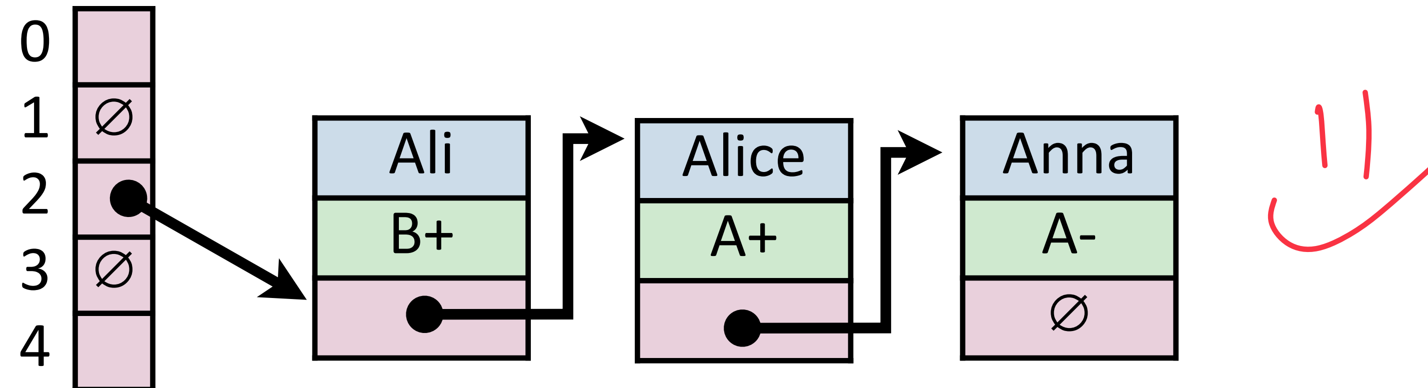
2. A data storage structure → array

3. **A method of addressing *hash collisions***

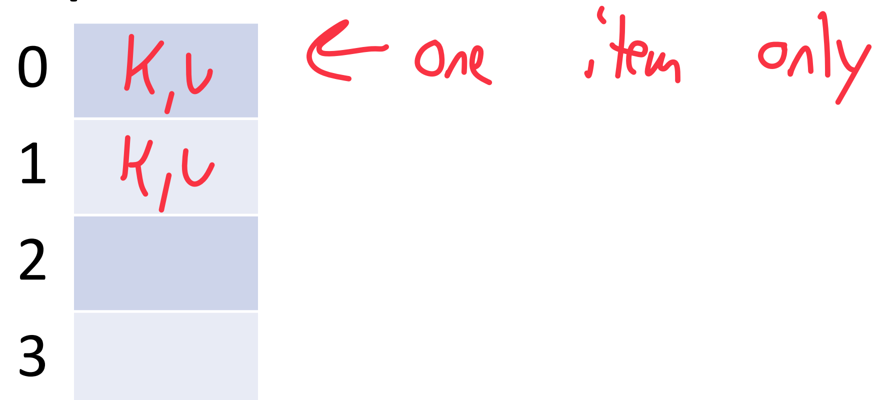
Open vs Closed Hashing

Addressing hash collisions depends on your storage structure.

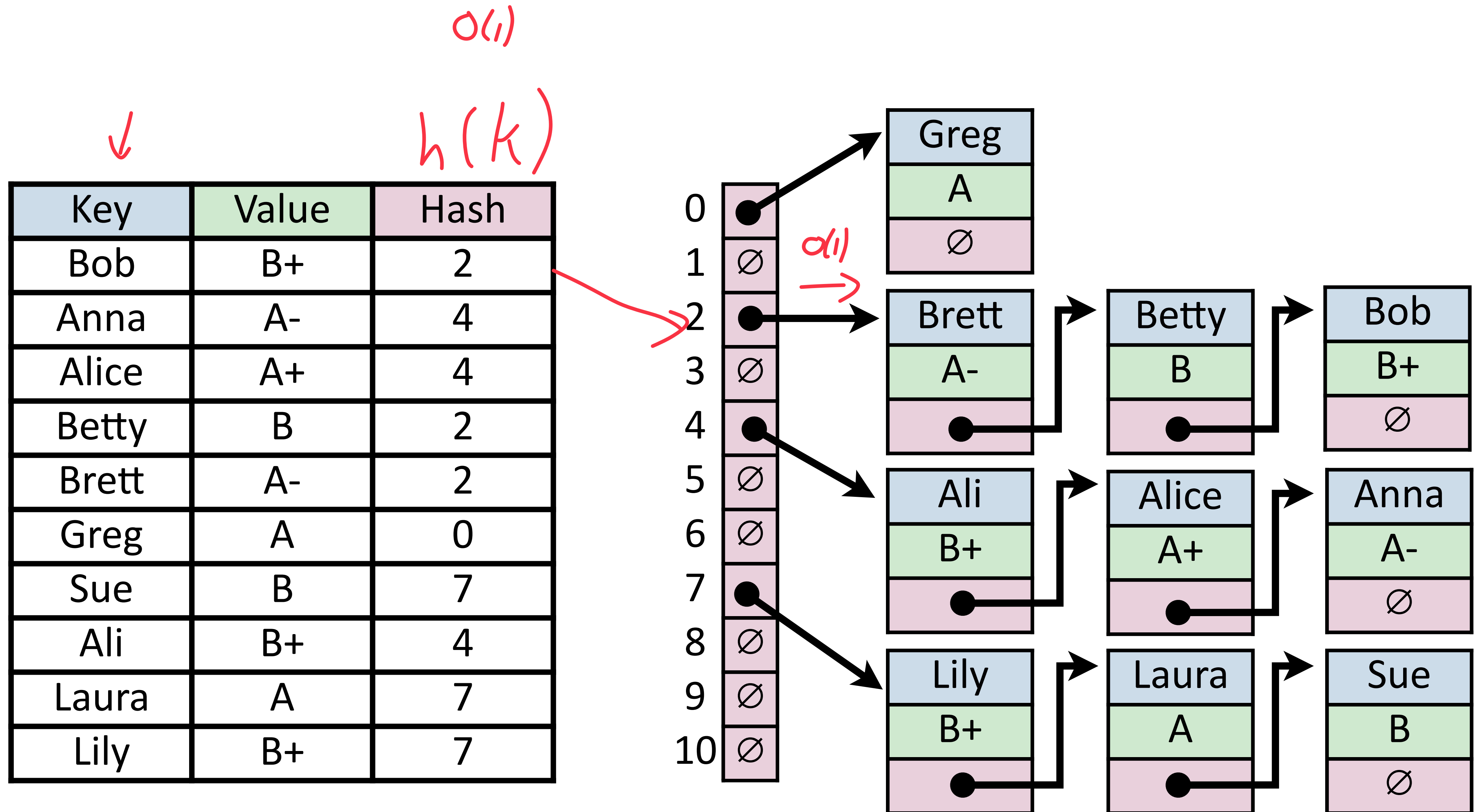
- **Open Hashing:** store *k,v* pairs externally



- **Closed Hashing:** store *k,v* pairs in the hash table



Hash Table (Separate Chaining)



Key	Value	Hash
Bob	B+	2
Anna	A-	4
Alice	A+	4
Betty	B	2
Brett	A-	2
Greg	A	0
Sue	B	7
Ali	B+	4
Laura	A	7
Lily	B+	7

Hash Table (Separate Chaining)

For hash table of size *m* and *n* elements:

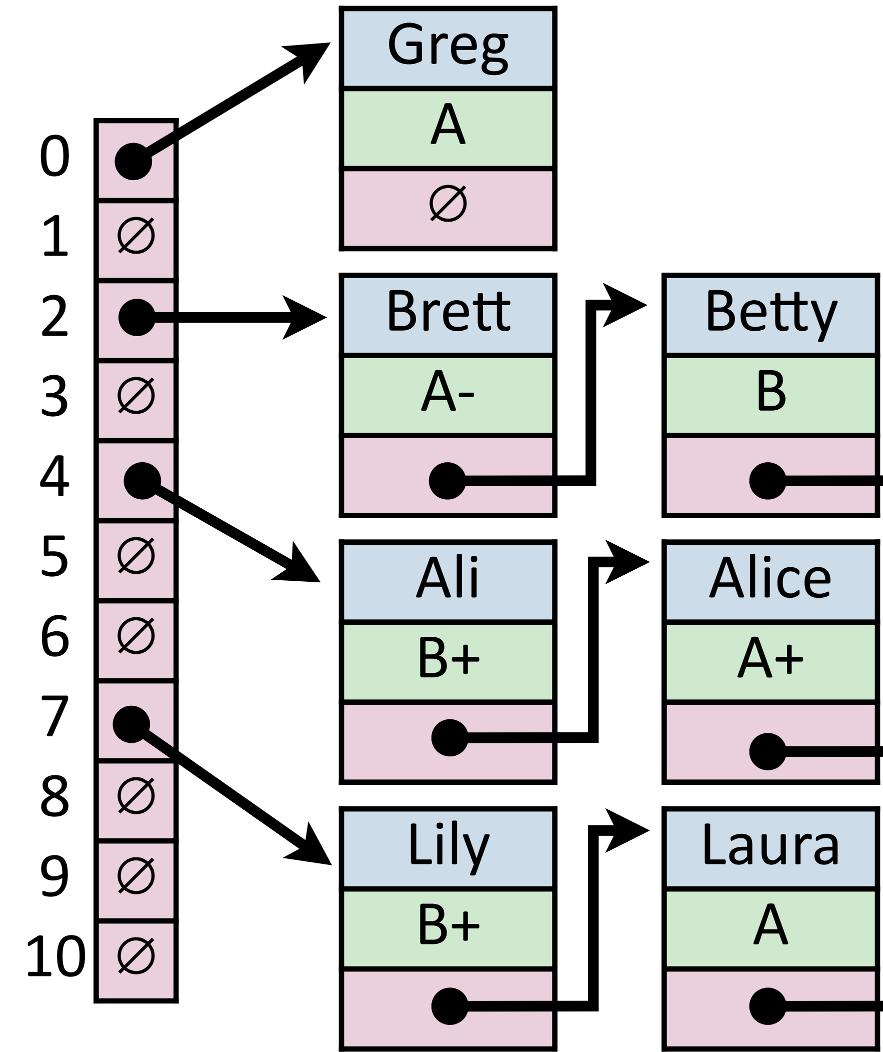
Find runs in: O(n)

Insert runs in: O(1)

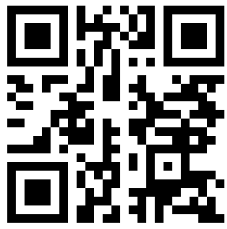
Remove runs in: O(n)

↳ Hash collisions

Pick hash s.t. all items hash here



Hash Table



Worst-Case behavior is bad — but what about randomness?

Where is the randomness in a hash table?

- input data random ← 40%
- select random hash function ← 33% ← (weak argument) but.... it works
- 1 hash functions assigns random values X
- size of array random X
- hash tables must be deterministic - No random! X
  ↳ True!

Hash Table

only give $h(x) = 1$ values

Worst-Case behavior is bad — but what about randomness?

1) **Fix *h***, our hash, and assume it is good for ***all keys***: (weak avg)

Simple uniform hashing assumption (SUHA)

↳ uniform independence

225 level

This is possible in real world!

2) Create a ***universal hash function family:***

Build precise set of hash functions

↳ Randomness is choice of hash !!

Hash Table

Worst-Case behavior is bad — but what about randomness?

1) **Fix *h***, our hash, and assume it is good for ***all keys***:

Simple Uniform Hashing Assumption

(Assume our dataset hashes optimally)

2) Create a ***universal hash function family:***

Given a collection of hash functions, pick one randomly

Like random quicksort if pick of hash is random, good expectation!

Simple Uniform Hashing Assumption

Given table of size m, a simple uniform hash, h, implies

2 keys — some chance of hashing to some value

$$\forall k_1, k_2 \in U \text{ where } k_1 \neq k_2,\ Pr(h[k_1] = h[k_2]) = \frac{1}{m}$$

Uniform: $Pr(h(k_1) = i) = \frac{1}{m}$ & $Pr(h(k_2) = i) = \frac{1}{m}$

↳ All positions in table equally likely for any input

Independent: $Pr(h[k_1] = i \mid h[k_2] = i) = Pr(h[k_1] = i) = \frac{1}{m}$

Simple Uniform Hashing Assumption

Given table of size m, a simple uniform hash, h, implies

$$\forall k_1, k_2 \in U \text{ where } k_1 \neq k_2,\ Pr(h[k_1] = h[k_2]) = \frac{1}{m}$$

Uniform: All keys equally likely to hash to any position

$$Pr(h[k_1]) = \frac{1}{m}$$

Independent: All key's hash values are independent of other keys

Separate Chaining Under SUHA

Table Size: m

Num objects: n

Claim: Under SUHA, expected length of chain is $\frac{n}{m}$

$\alpha_j =$ expected # of items hashing to position j

$$\alpha_j = \sum_i H_{i,j}$$

$$H_{i,j} = \begin{cases} 1 \text{ if item i hashes to j} \\ 0 \text{ otherwise} \end{cases}$$

Separate Chaining Under SUHA

Table Size: m

Num objects: n

Claim: Under SUHA, expected length of chain is $\frac{n}{m}$

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$$\alpha_j = \sum_i H_{i,j}$$

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$$E[\alpha_j] = E[\sum_i H_{i,j}]$$

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$$E[\alpha_j] = E[\sum_i H_{i,j}]$$

$$E[\alpha_j] = \sum_i Pr(H_{i,j} = 1) * 1 + Pr(H_{i,j} = 0) * 0$$

Separate Chaining Under SUHA

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$$E[\alpha_j] = \sum_i Pr(H_{i,j} = 1) * 1 + Pr(H_{i,j} = 0) * 0$$

$$E[\alpha_j] = n * Pr(H_{i,j} = 1)$$

Separate Chaining Under SUHA

Table Size: m

Num objects: n

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$$E[\alpha_j] = E[\sum_i H_{i,j}]$$

$$Pr[H_{i,j} = 1] = \frac{1}{m}$$

$$E[\alpha_j] = n * Pr(H_{i,j} = 1)$$

Separate Chaining Under SUHA

Claim: Under SUHA, expected length of chain is $\frac{n}{m}$

Table Size: m

Num objects: n

$\alpha_j =$ expected # of items hashing to position j

$$\alpha_j = \sum_i H_{i,j}$$

$$H_{i,j} = \begin{cases} 1 \text{ if item i hashes to j} \\ 0 \text{ otherwise} \end{cases}$$

linearity of expectation

$$E[\alpha_j] = E[\sum_i H_{i,j}]$$

$$Pr[H_{i,j} = 1] = \frac{1}{m}$$

drop $Pr(H_{i,j} = 0)$

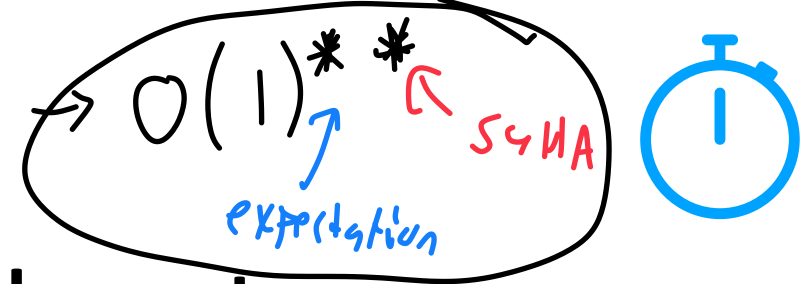
$$E[\alpha_j] = n * Pr(H_{i,j} = 1)$$

plug in SUHA

α = Load factor = $\frac{\text{\# items}}{\text{\# positions}}$

$$\mathbf{E[\alpha_j] = \frac{n}{m}}$$

Separate Chaining Under SUHA

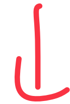


Under SUHA, a hash table of size *m* and *n* elements:

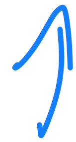
Find runs in: $1 + \alpha$.

You tell me n

I set m myself



α is a constant

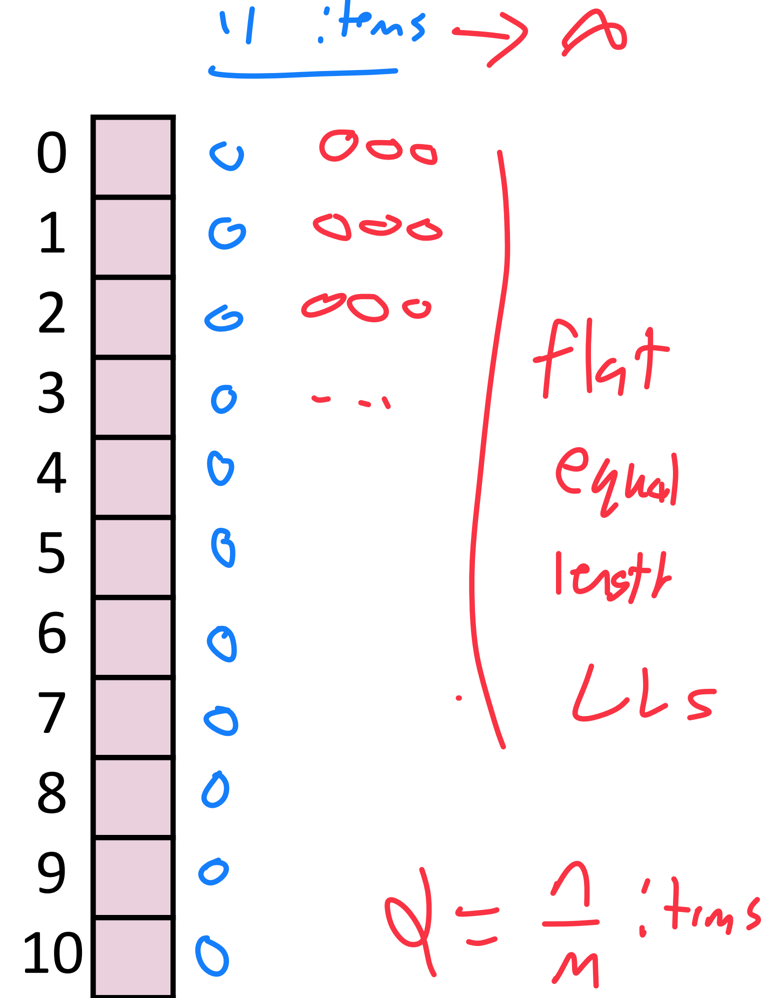


literal tradeoff!

Insert runs in: $O(1)$.

Remove runs in: $1 + \alpha$.

O(1) lookup LL lookup



n items → ∞

$\alpha = \frac{n}{m}$ items

(Example of closed hashing)

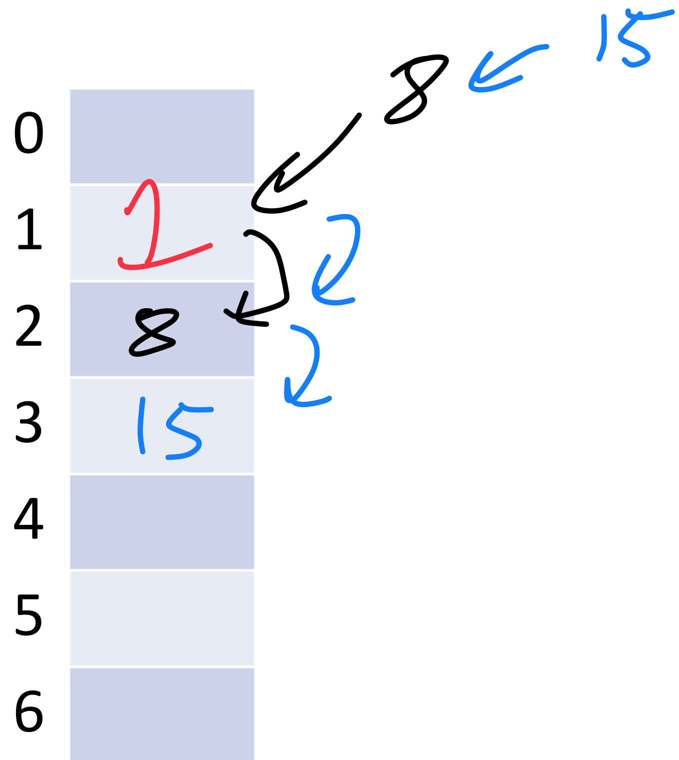
Collision Handling: Probe-based Hashing

S = { 1, 8 , 15} → same value 1

h(k) = k % 7

|S| = n

|Array| = m



1) Hash item

2) Insert at hash value

If not possible...

Pick another space

↳ easiest is just +1

(Example of closed hashing)

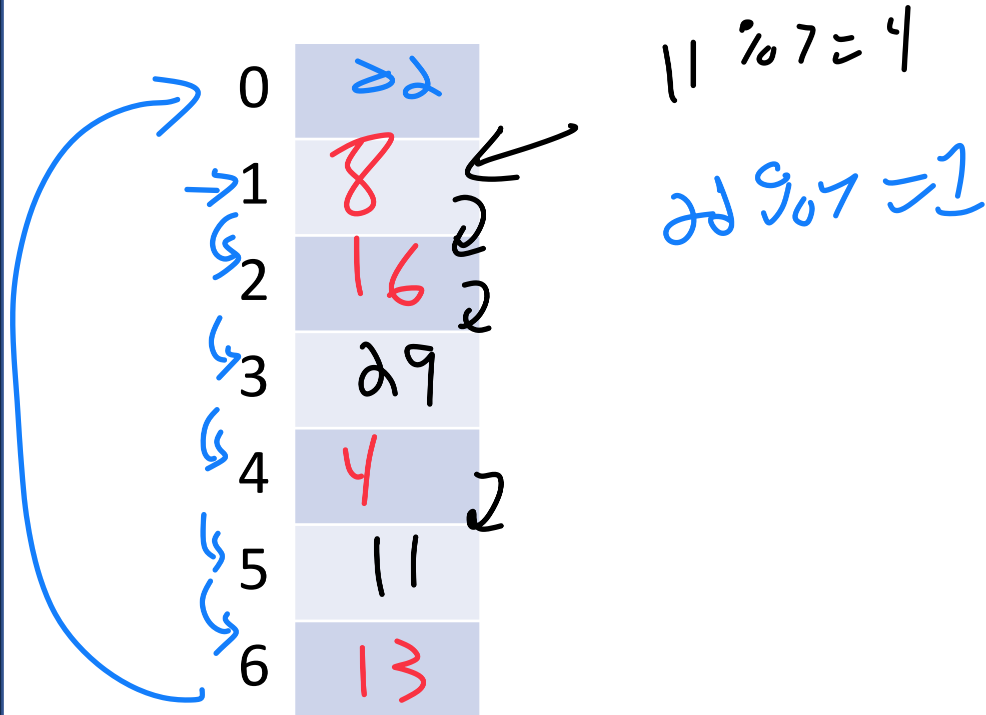
Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 }

|S| = n

h(k) = k % 7

|Array| = m



h(k, i) = (k + i) % 7

Try h(k) = (k + 0) % 7, if full…

Try h(k) = (k + 1) % 7, if full…

Try h(k) = (k + 2) % 7, if full…

Try …

(Example of closed hashing)

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 } **|S| = n**

h(k, i) = (k + i) % 7 **|Array| = m**

0	22
1	8
2	16
3	29
4	4
5	11
6	13

`_find(29)`

1) Hash input 29 % 7 = 1

2) If not match, add +1 and repeat

Stop when:

A) we find item of interest

B) we loop back to initial address

C) If we find empty position

(Example of closed hashing)

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 } **|S| = n**

h(k, i) = (k + i) % 7 **|Array| = m**

0	22
1	8
2	16
3	29
4	4
5	11
6	13

`_find(29)`

1) Hash the input key [h(29)=1]

2) Look at hash value (address) position

   If present, return (k, v)

   If not look at **next available space** → i+1

Stop when:

1) We find the object we are looking for
2) We have searched every position in the array
3) We find a blank space w/o a tombstone value

(Example of closed hashing)

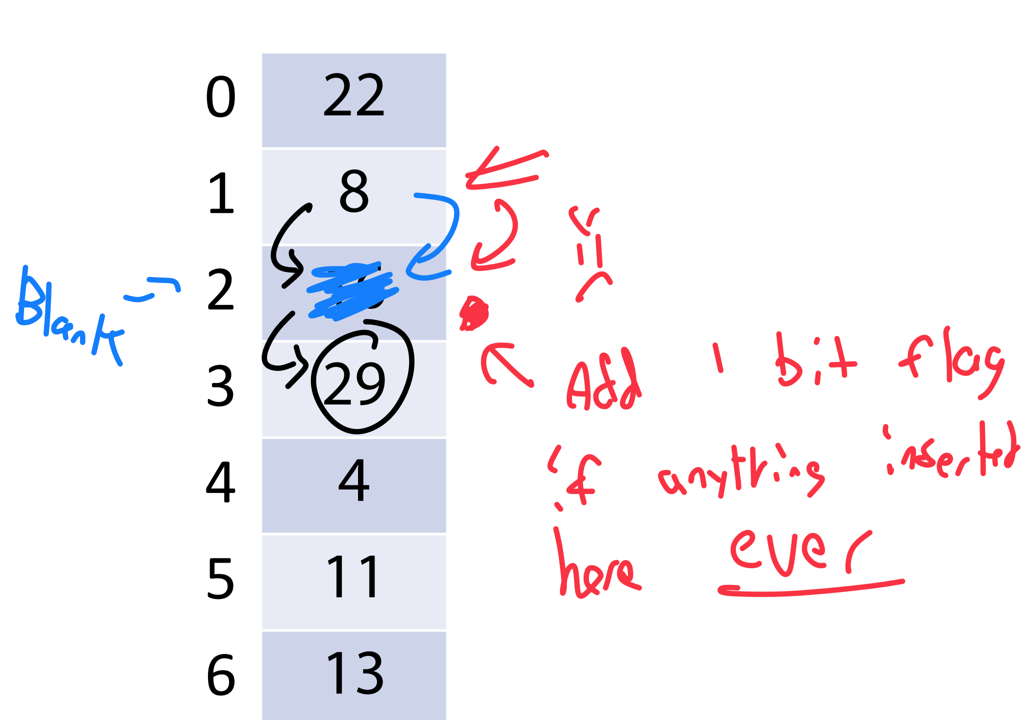
Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 }

|S| = n

h(k, i) = (k + i) % 7

|Array| = m



_remove(16)

1) Find (16)

2) Remove key, value pair

↳ Tombstone position

_find(29)

(Example of closed hashing)

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 }

|S| = n

h(k, i) = (k + i) % 7

|Array| = m

0	22
1	8
2	16
3	29
4	4
5	11
6	13

`_remove(16)`

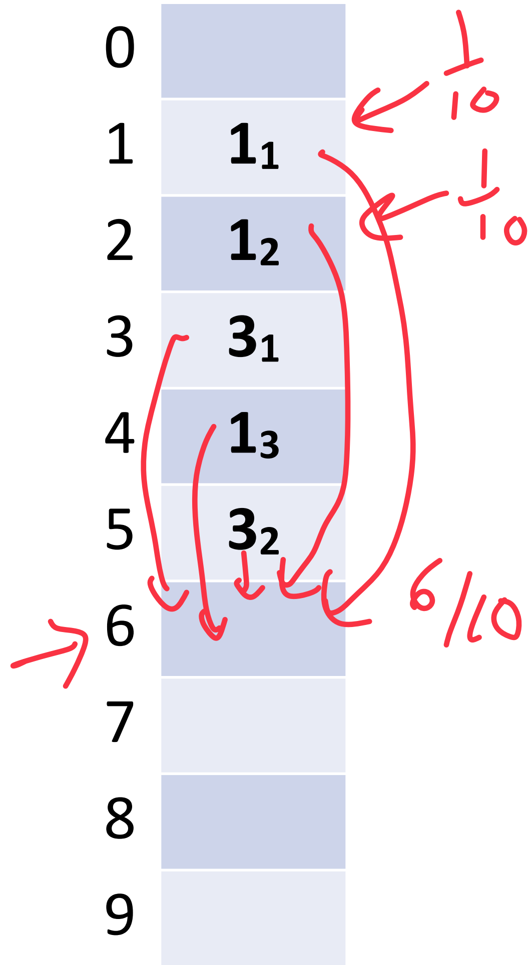
1) Hash the input key [h(16)=2]
2) Find the actual location (if it exists)
3) Remove the (k,v) at hash value (address)

Don't resize the array! Tombstone!

A Problem w/ Linear Probing

Primary Clustering:

Ideal hash is SUHA ⟨ uniform, independence



Description: Linear probing leads to large clusters of k,v pairs

"rich get richer"

↳ This is not uniform in practice!

↳ More clusters == more collisions

Remedy:

↳ We need to pick a better next available!

(Example of closed hashing)

Collision Handling: Quadratic Probing

S = { 16, 8, 4, 13, 29, 12, 22 }

|S| = n

h(k) = k % 7

|Array| = m

0	
1	8
2	16
3	
4	4
5	
6	13

h(k, i) = (k + i*i) % 7

Try h(k) = (k + 0) % 7, if full…

Try h(k) = (k + 1*1) % 7, if full…

Try h(k) = (k + 2*2) % 7, if full…

Try …

9

16

23

⋮

A Problem w/ Quadratic Probing

Secondary Clustering:

0	0_1
1	0_2
2	
3	
4	0_3
5	
6	
7	
8	
9	0_4

Description:

Remedy:

(Example of closed hashing)

Collision Handling: Double Hashing

S = { 16, 8, 4, 13, 29, 11, 22 }

$h_1(k) = k \% 7$

$h_2(k) = 5 - (k \% 5)$

$|S| = n$

$|Array| = m$

0	
1	**8**
2	**16**
3	
4	**4**
5	
6	**13**

$h(k, i) = (h_1(k) + i*h_2(k)) \% 7$

Try $h(k) = (k + 0*h_2(k)) \% 7$, if full...

Try $h(k) = (k + 1*h_2(k)) \% 7$, if full...

Try $h(k) = (k + 2*h_2(k)) \% 7$, if full...

Try ...

Running Times

(Don't memorize these equations, no need.)

(Expectation under SUHA)

Open Hashing:

insert: ___________.

find/ remove: ___________.

Closed Hashing:

insert: ___________.

find/ remove: ___________.

Running Times (Expectation under SUHA)

Open Hashing: $0 \leq \alpha \leq \infty$

insert: 1 .

find/ remove: $1 + \alpha$.

Closed Hashing: $0 \leq \alpha < 1$

insert: $\frac{1}{1-\alpha}$.

find/ remove: $\frac{1}{1-\alpha}$.

Observe:

- As α increases:

Runtime approaches infinity

- If α is constant:

Runtime is constant

Running Times *(Don't memorize these equations, no need.)*

The expected number of probes for find(key) under SUHA

Linear Probing:

- Successful: **$\frac{1}{2}(1 + 1/(1-\alpha))$**
- Unsuccessful: **$\frac{1}{2}(1 + 1/(1-\alpha))^2$**

Double Hashing:

- Successful: **$1/\alpha * \ln(1/(1-\alpha))$**
- Unsuccessful: **$1/(1-\alpha)$**

Separate Chaining:

- Successful: **$1 + \alpha/2$**
- Unsuccessful: **$1 + \alpha$**

Instead, observe:

- As α increases:

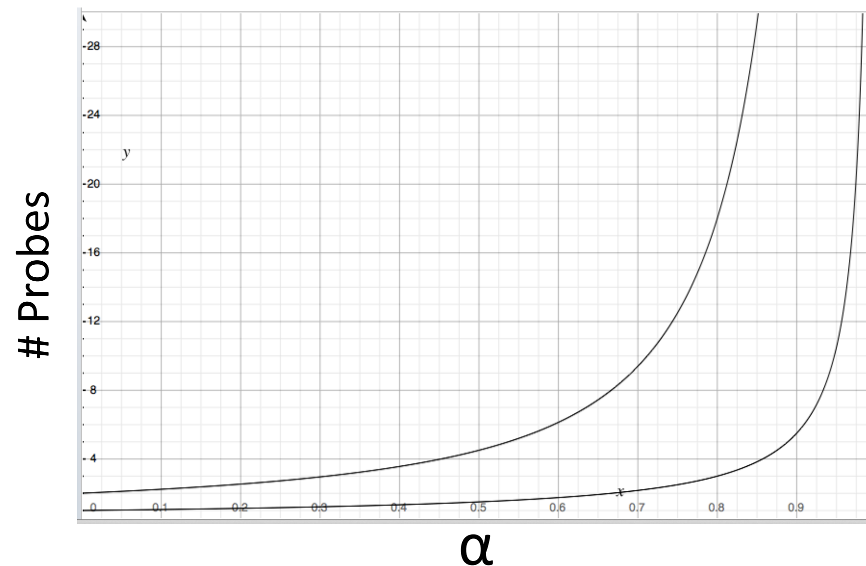
- If α is constant:

Running Times

The expected number of probes for find(key) under SUHA

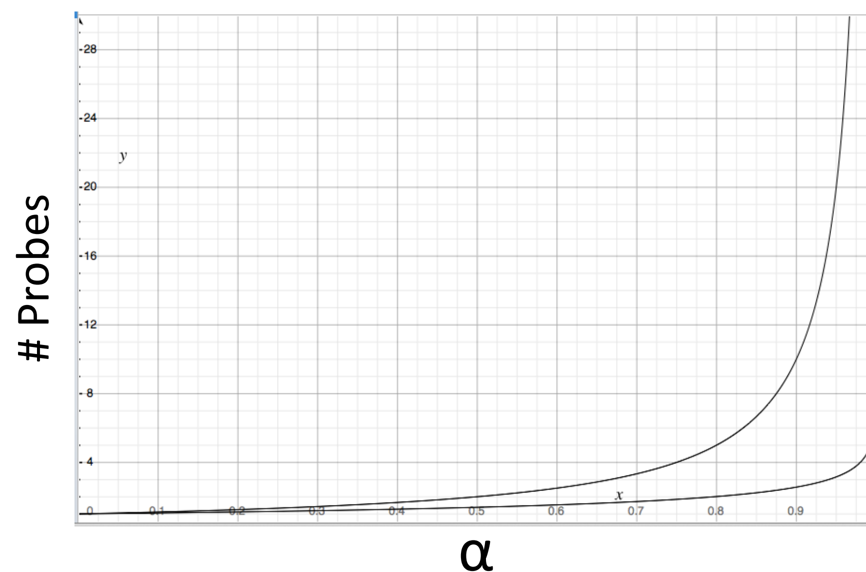
Linear Probing:

- Successful: **½(1 + 1/(1-α))**
- Unsuccessful: **½(1 + 1/(1-α))²**



Double Hashing:

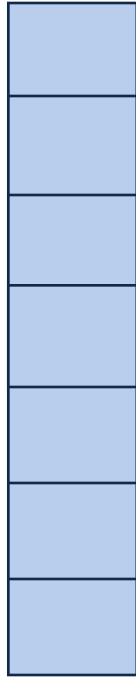
- Successful: **1/α * ln(1/(1-α))**
- Unsuccessful: **1/(1-α)**



When do we resize?

Resizing a hash table

How do you resize?



Which collision resolution strategy is better?

- Big Records:
- Structure Speed:

What structure do hash tables implement?

What constraint exists on hashing that doesn’t exist with BSTs?

Why talk about BSTs at all?

std::map in C++

```
T& map<K, V>::operator[]
pair<iterator, bool> map<K, V>::insert()
iterator map<K, V>::erase()


iterator map<K, V>::lower_bound( const K & );
iterator map<K, V>::upper_bound( const K & );
```

std::unordered_map in C++

```
T& unordered_map<K, V>::operator[]
pair<iterator, bool> unordered_map<K, V>::insert()
iterator unordered_map<K, V>::erase()
```

~~`iterator map<K, V>::lower_bound( const K & );`~~

~~`iterator map<K, V>::upper_bound( const K & );`~~

```
float unordered_map<K, V>::load_factor();
void unordered_map<K, V>::max_load_factor(float m);
```

Running Times

	Hash Table	AVL	Linked List
Find	Expectation*: Worst Case:		
Insert	Expectation*: Worst Case:		
Storage Space			