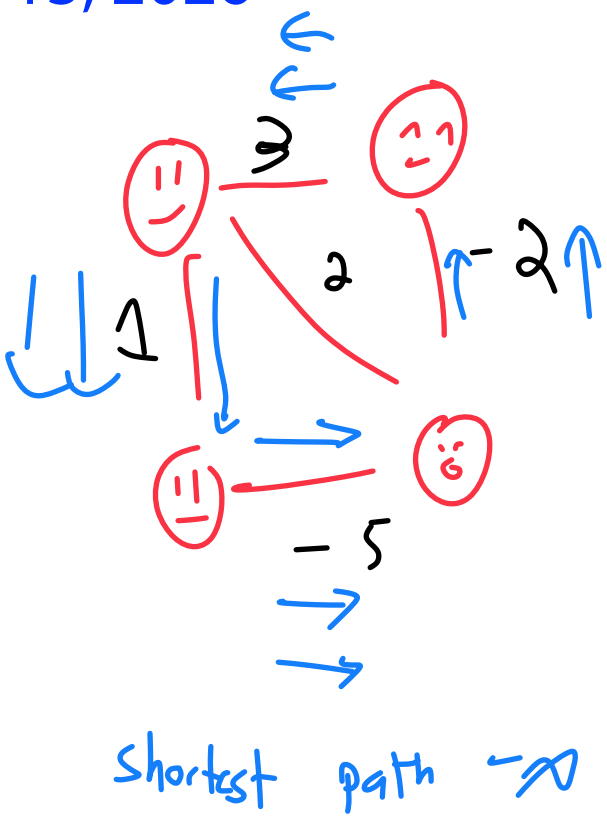
Data Structures and Algorithms

SSSP and All Paths Shortest Path

CS 225

Brad Solomon

April 13, 2026



UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Department of Computer Science

Learning Objectives

Finalize Dijkstra's Algorithm implementation

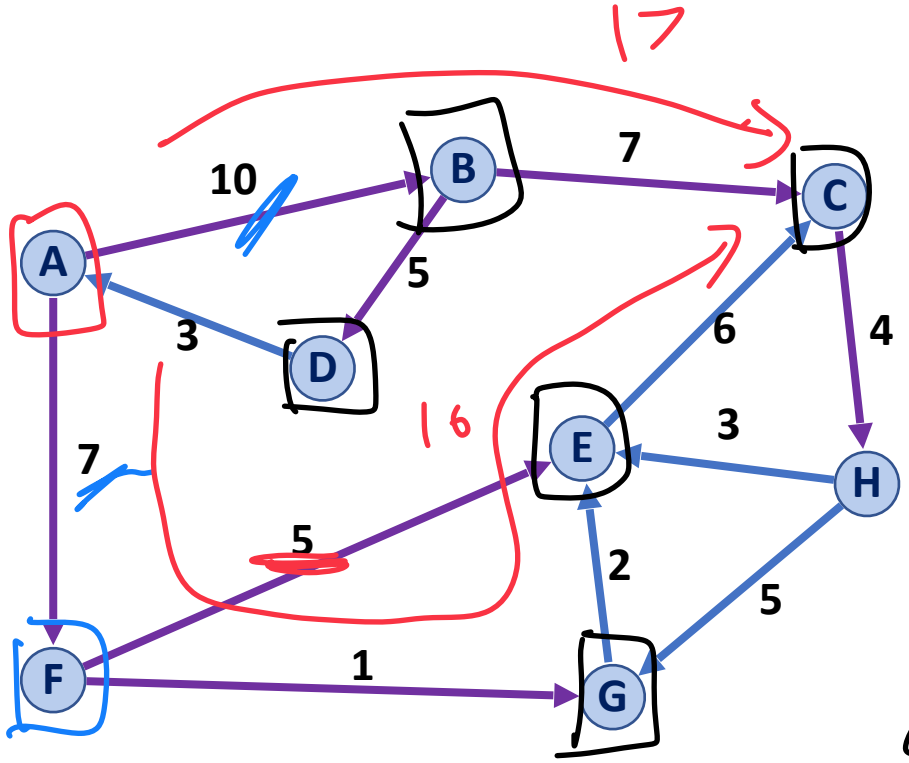
Introduce and discuss All-Pairs Shortest Path

See some probability?

MP Puzzle

↳ A* Search

Dijkstra's Algorithm (SSSP)

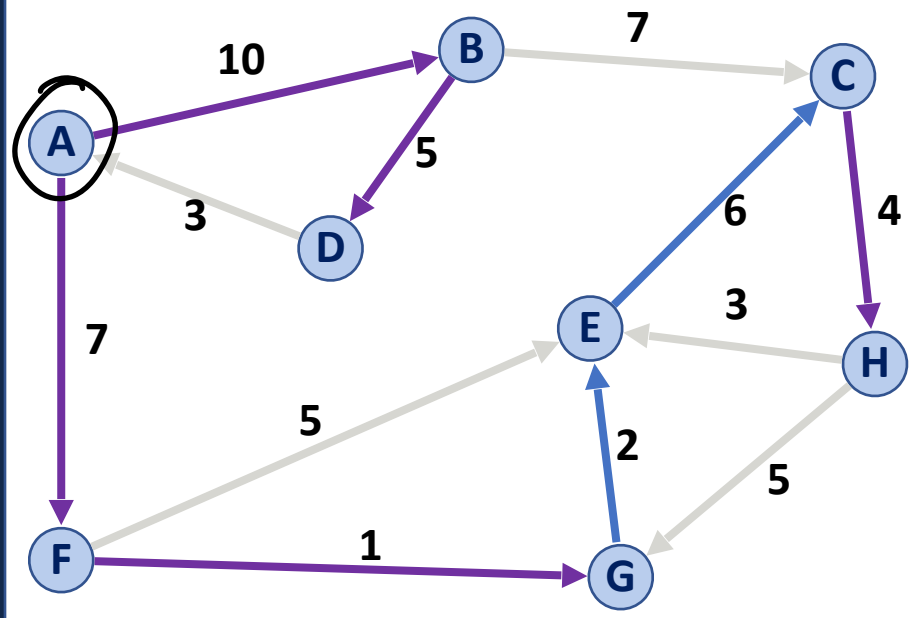



```
DijkstraSSSP(G, s):
  foreach (Vertex v : G.vertices()):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q // min distance, defined by d[v]
  Q.buildHeap(G.vertices())
  Graph T          // "labeled set"

  repeat n times:
    Vertex u = Q.removeMin()
    T.add(u)
    foreach (Vertex v : neighbors of u not in T):
      if cost(u, v) + d[u] < d[v]:
        d[v] = cost(u, v) + d[u]
        p[v] = u
```

A	B	C	D	E	F	G	H
--	A	E	B	G	A	F	C
0	10	16	15	10	7	8	20

Dijkstra's Algorithm (SSSP)

← this does not solve MST!



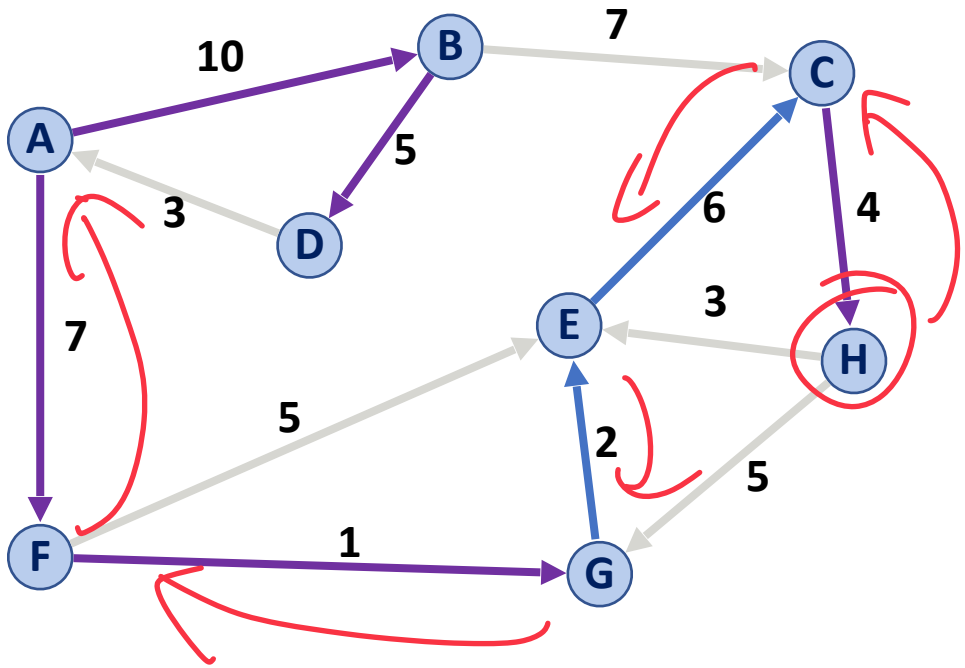

```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G.vertices()):
 7      d[v] = +inf
 8      p[v] = NULL
 9    d[s] = 0
10
11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T           // "labeled set"
14
15    repeat n times:
16      Vertex u = Q.removeMin()
17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
21          p[v] = u
```

↑ Now you solve MST

(↑ different weight factors)

A	B	C	D	E	F	G	H
--	A	E	B	G	A	F	C
0	10	16	15	10	7	8	20

Dijkstra's Algorithm (SSSP)



Whats the point of predecessor?

↳ Get dist of path easily

↳ To get path, use predecessor

	A	B	C	D	E	F	G	H
P	--	A	E	B	G	A	F	C
D	0	10	16	15	10	7	8	20

Dijkstra's Algorithm (SSSP)

What is the running time of Dijkstra's Algorithm?

```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G):
 7      d[v] = +inf
 8      p[v] = NULL
 9    d[s] = 0
10
11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T          // "labeled set"
14
15    repeat n times:
16      Vertex u = Q.removeMin()
17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
21          p[v] = m
22
23    return T
```

Dijkstra's Algorithm (SSSP)

What is the running time of Dijkstra's Algorithm?

O(m + n log n)

The same as Prim's!

(Times here are minheap)

6-9: O(n)

11-12: O(n)

15: repeat below n x

16-22: O(log n)

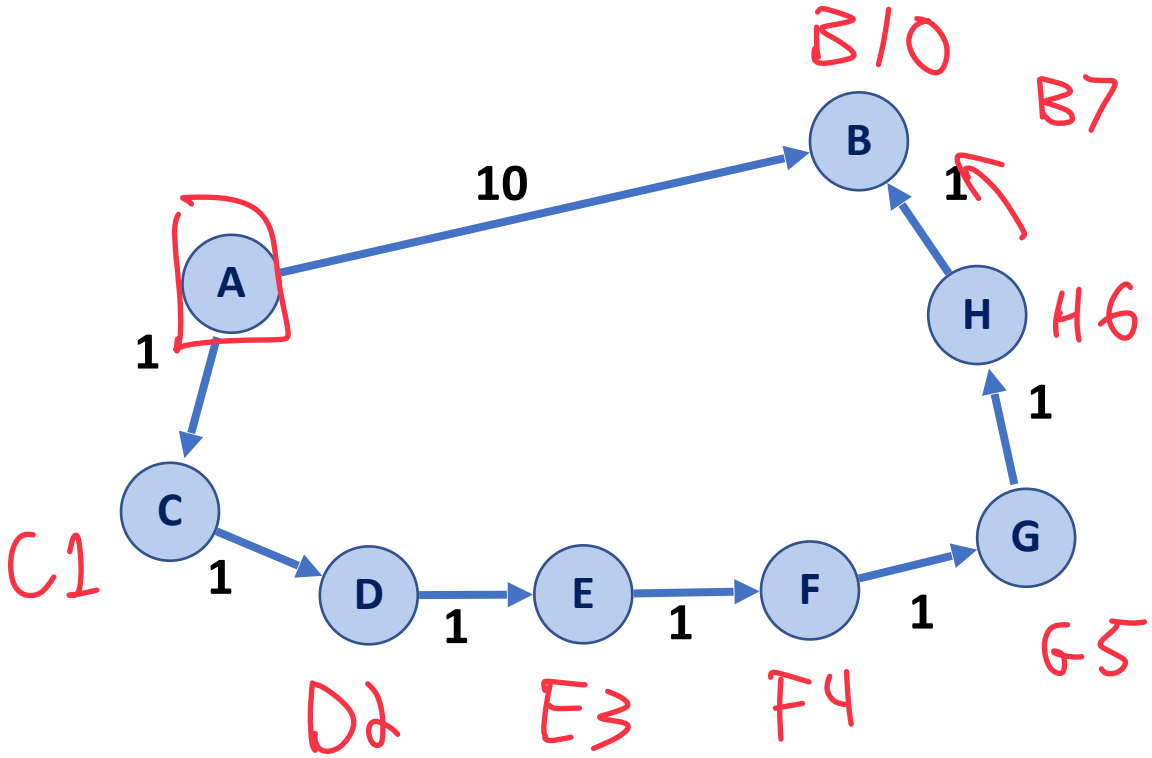
[w/ Fib Heap O(1) updates]

```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G):
 7      d[v] = +inf
 8      p[v] = NULL
 9    d[s] = 0
10
11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T          // "labeled set"
14
15    repeat n times:
16      Vertex u = Q.removeMin()
17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
21          p[v] = m
22
23    return T
```

Dijkstra's Algorithm (SSSP)

Claim: Dijkstras will always visit a node through its optimal shortest path.

When we will visit B in the following graph?



A → 10 → B

A → _ → B ?

If A → _ → B is smaller than A → B

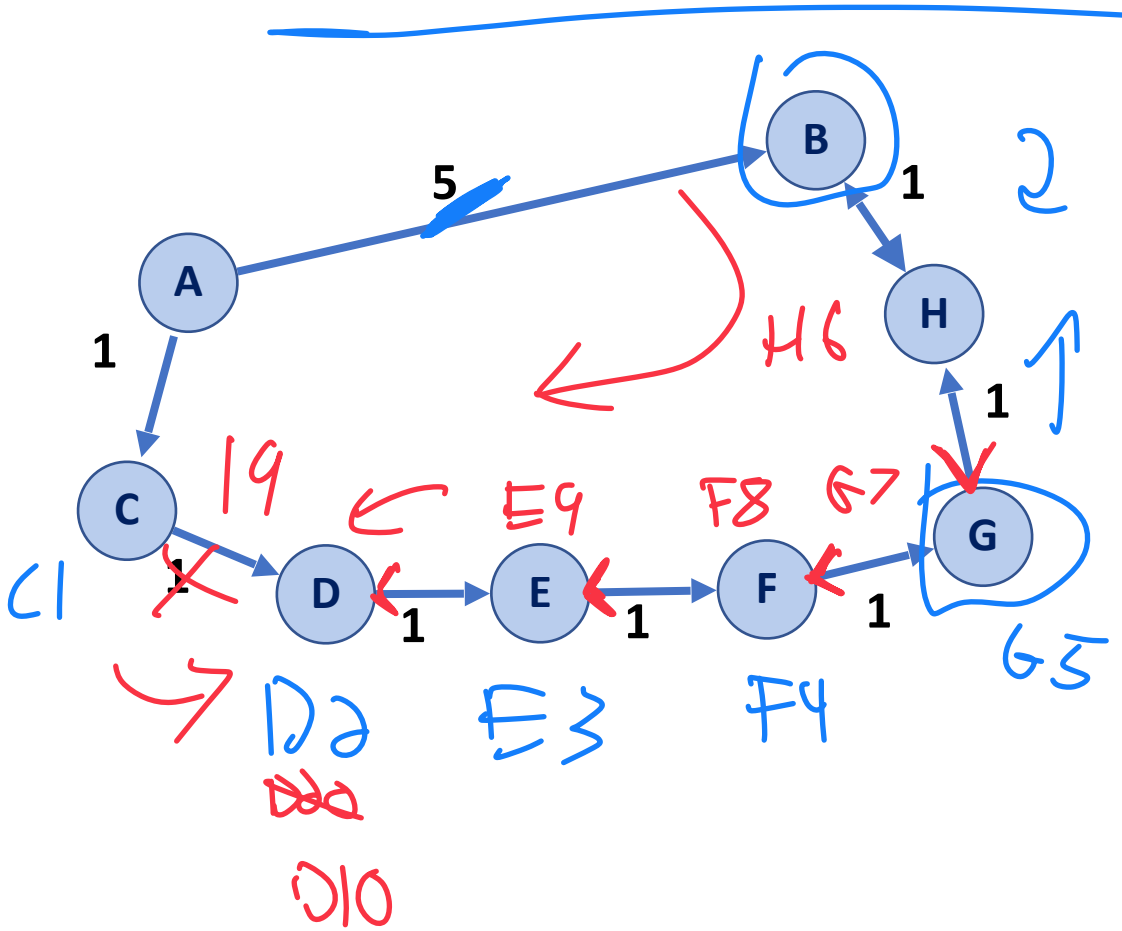
Then A → _ must be smaller too.

If A → _ is smaller, we will visit it first

Dijkstra's Algorithm (SSSP)

Claim: Dijkstras will always visit a node through its optimal shortest path.

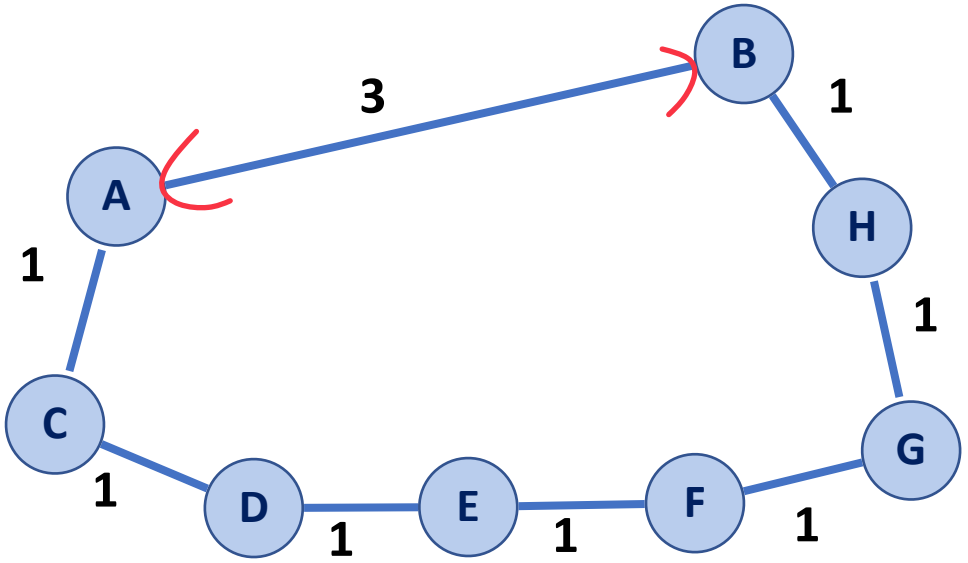
When we will visit H in the following graph?



either! whichever is in priority Queue first

Dijkstra's Algorithm (SSSP)

How does Dijkstra's algorithm handle undirected graphs?

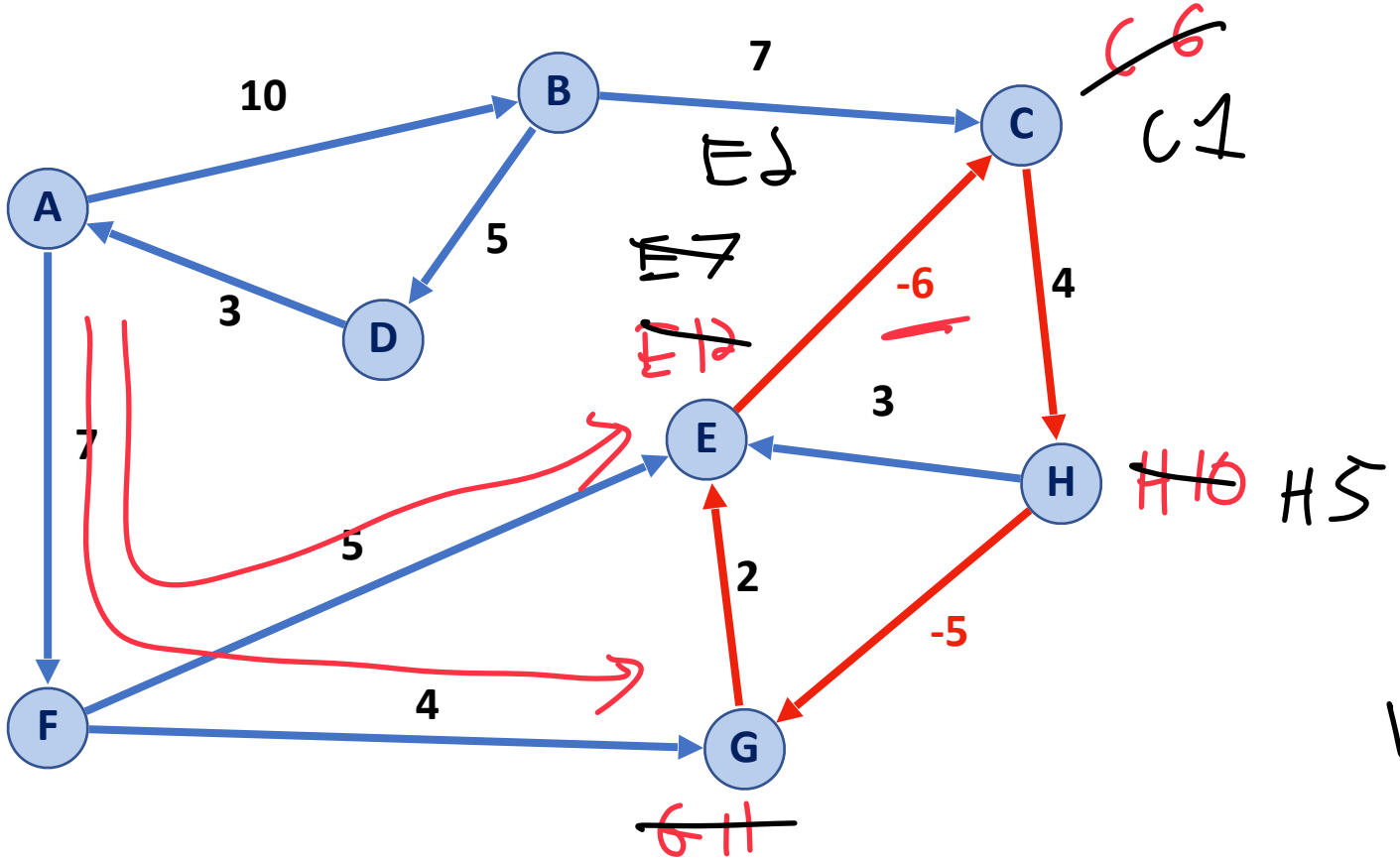


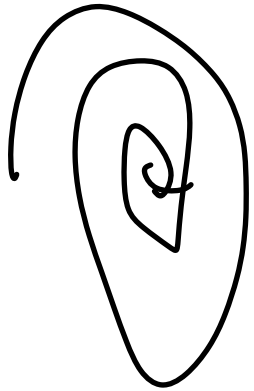
undirected is fine

Just treat every edge as both directions!

Dijkstra's Algorithm (SSSP)

How does Dijkstras handle a negative weight cycle?

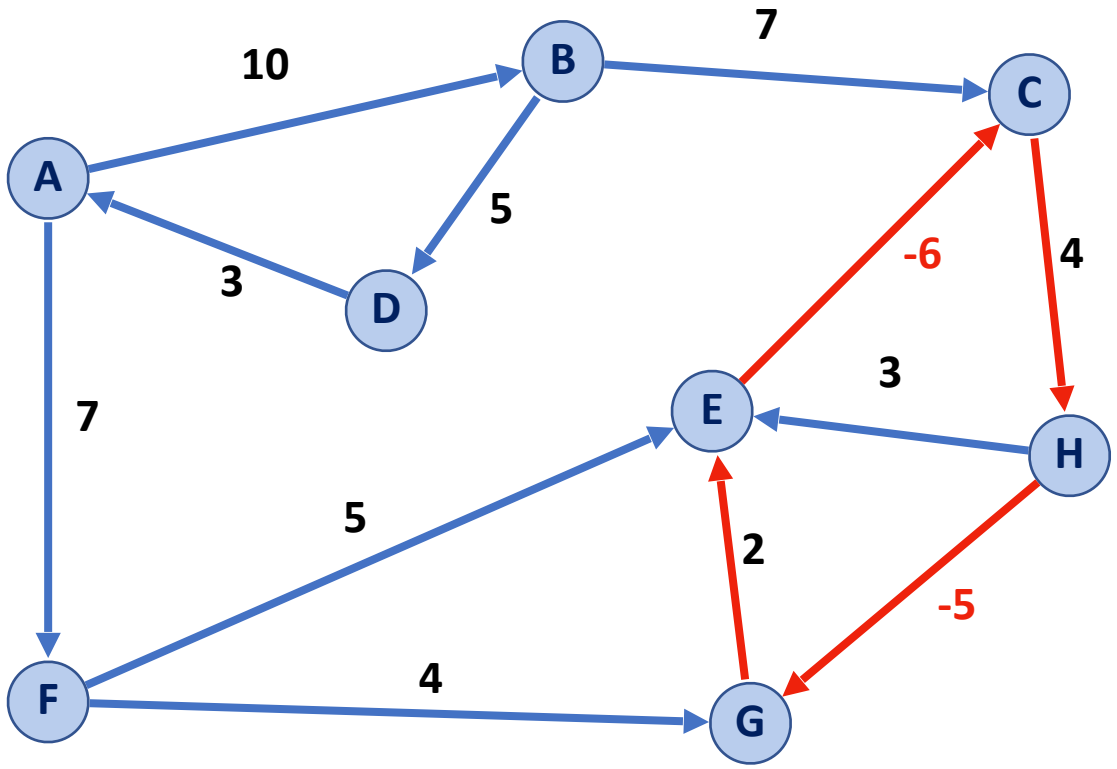




Will infinite loop is $-\infty$

Dijkstra's Algorithm (SSSP)

How does Dijkstras handle a negative weight cycle?



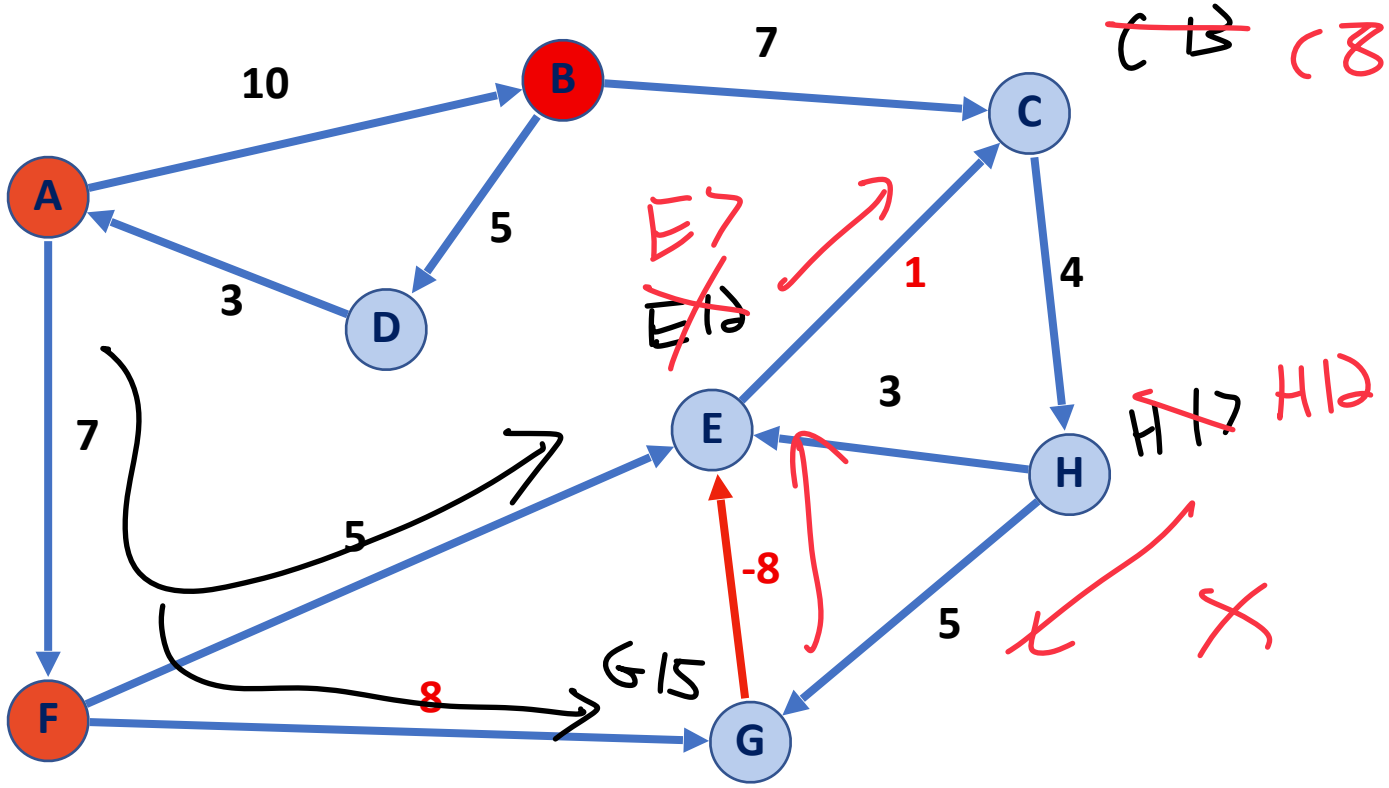
Shortest Path (A → E): A → F → E → (C → H → G → E)*

Length: 12

Length: -5 (repeatable)

Dijkstra's Algorithm (SSSP)

How does Dijkstras handle a negative weight edge without a cycle?



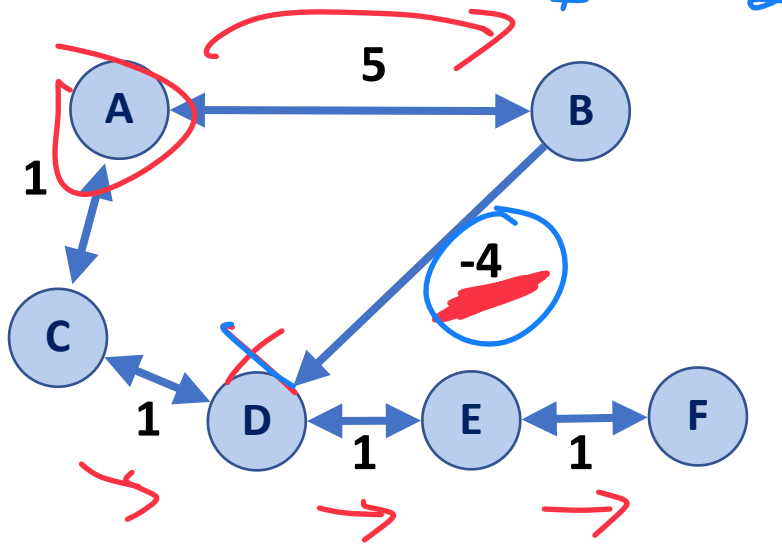
A	B	C	D	E	F	G	H
--	A	B	B	F	A	F	--
0	10	17	15	12	7	15	∞

Dijkstra's Algorithm (SSSP)

We assume that item pulled out of priority queue is **the next smallest item**

Negative weights break this assumption!

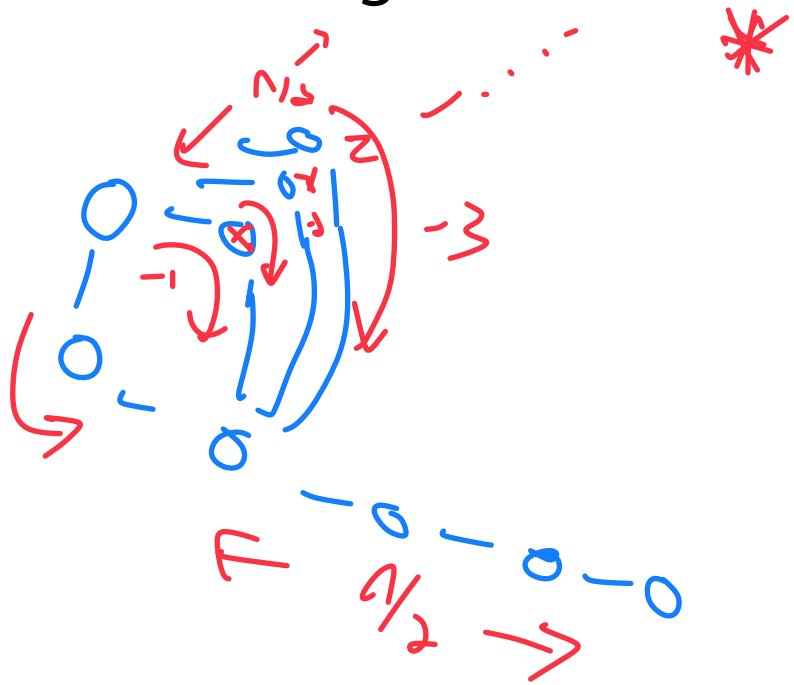
A	B	C	D	E	F
--	A	A	~~C~~ B	D	E
0	5	2	~~2~~ 1	~~3~~ 2	~~4~~ 3



Dijkstra's Algorithm (SSSP)

Tangent point this & next slide

Recalculating all distances is possible, but algorithm runtime is very bad!

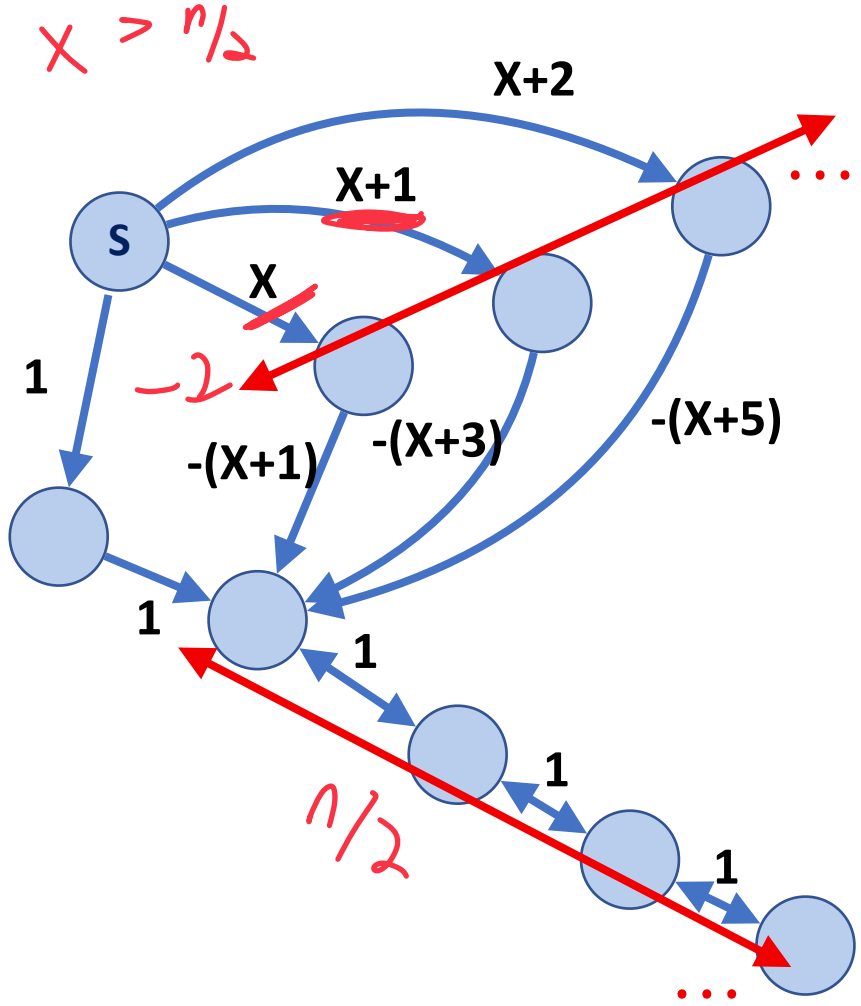


```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G):
 7      d[v] = +inf
 8      p[v] = NULL
 9    d[s] = 0
10
11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T          // "labeled set"
14
15    repeat until Q.empty():
16      Vertex u = Q.removeMin()
17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
21          p[v] = m
22          if v not in Q:
23             Q.push(v)
24    return T
```

Dijkstra's Algorithm (SSSP)

Recalculating all distances is possible, but algorithm runtime is very bad!

Worst case: ~n/2 nodes each updating ~n/2 nodes distances




```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G):
 7      d[v] = +inf
 8      p[v] = NULL
 9    d[s] = 0
10
11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T          // "labeled set"
14
15    repeat until Q.empty():
16      Vertex u = Q.removeMin()
17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
21          p[v] = m
22          if v not in Q:
23             Q.push(v)
24    return T
```

Dijkstra's Algorithm (SSSP)

Dijkstras Algorithm works only on non-negative weights

Optimal implementation:

Fibonacci Heap

If dense, unsorted list ties

Optimal runtime:

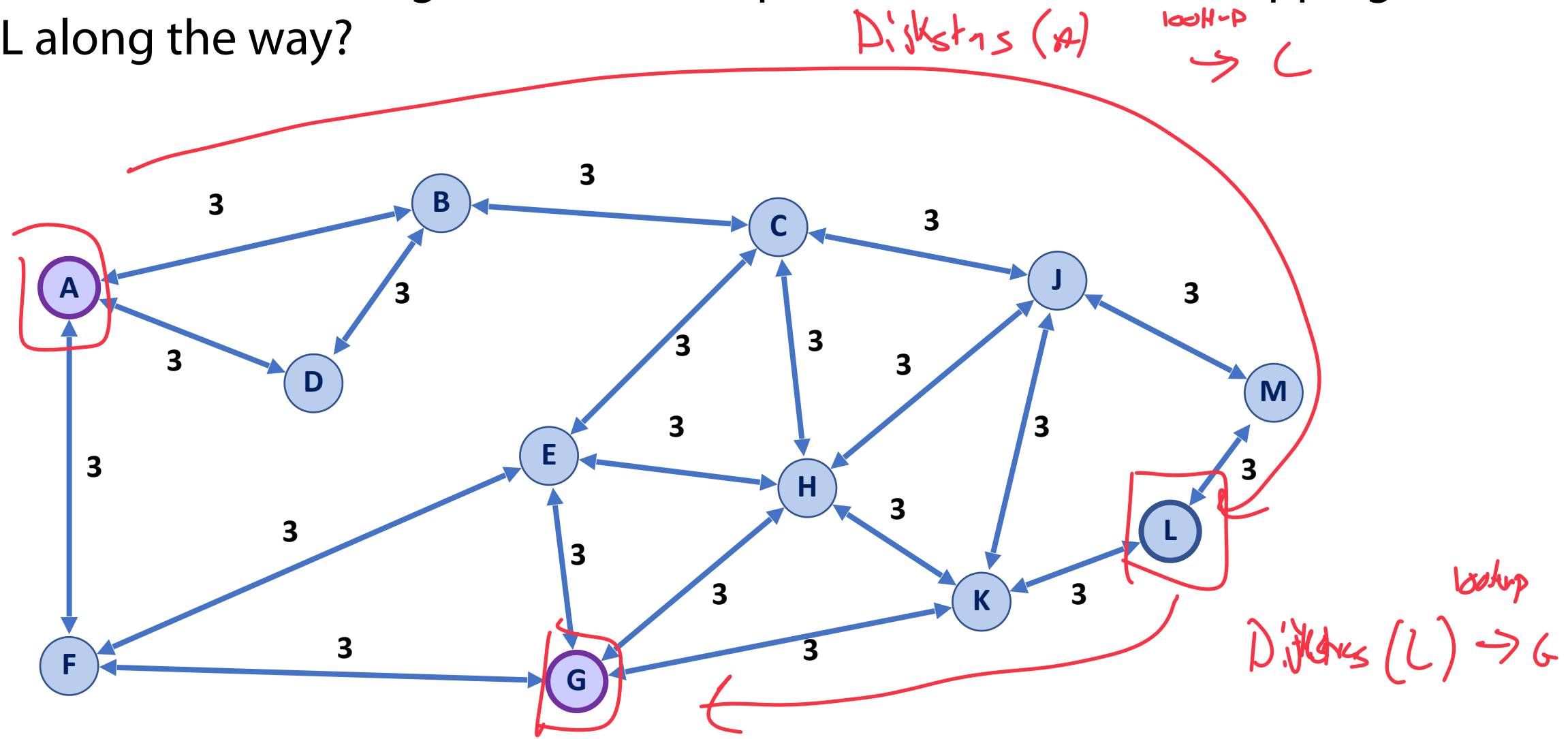
Sparse: $O(m + n \log n)$

Dense: $O(n^2)$

```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G):
 7      d[v] = +inf
 8      p[v] = NULL
 9    d[s] = 0
10
11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T          // "labeled set"
14
15    repeat n times:
16      Vertex u = Q.removeMin()
17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
21          p[v] = m
22
23    return T
```

Landmark Path Problem

What if I wanted to get the shortest path from A to G but stopping at L along the way?



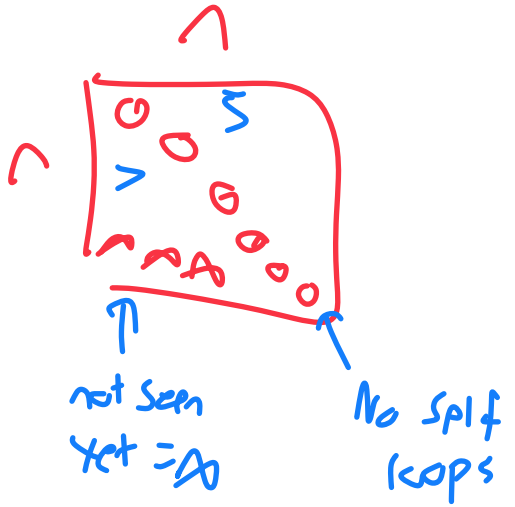
Floyd-Warshall Algorithm

Floyd-Warshall's Algorithm is an alternative to Dijkstra in the presence of negative-weight edges (not negative weight cycles).

```
 1  FloydWarshall(G):
 2    Let d be a adj. matrix initialized to +inf
 3    foreach (Vertex v : G):
 4      d[v][v] = 0
 5    foreach (Edge (u, v) : G):
 6      d[u][v] = cost(u, v)
 7
 8    foreach (Vertex u : G):
 9      foreach (Vertex v : G):
10        foreach (Vertex w : G):
11          if (d[u, v] > d[u, w] + d[w, v])
12            d[u, v] = d[u, w] + d[w, v]
```

weights of edges we know

Initialize



u → v

vs

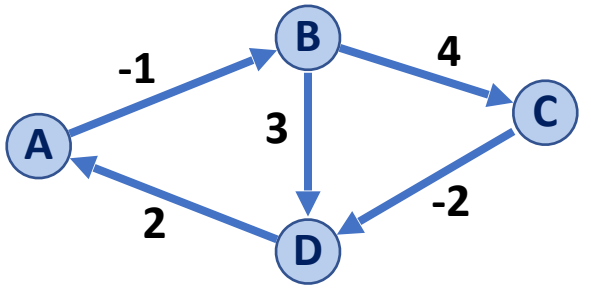
u → w → v

which is better?

Floyd-Warshall Algorithm

```
1  FloydWarshall(G):
2    Let d be a adj. matrix initialized to +inf
3    foreach (Vertex v : G):
4      d[v][v] = 0
5    foreach (Edge (u, v) : G):
6      d[u][v] = cost(u, v)
```

	A	B	C	D
A	0	-1	∞	∞
B	∞	0	4	3
C	∞	∞	0	-2
D	2	∞	∞	0

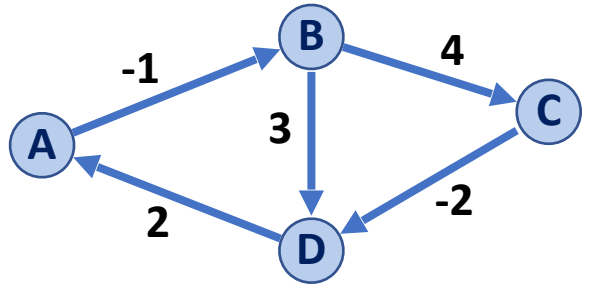


Floyd-Warshall Algorithm

```
 8    foreach (Vertex w : G):
 9      foreach (Vertex u : G):
10        foreach (Vertex v : G):
11          if (d[u, v] > d[u, w] + d[w, v])
12            d[u, v] = d[u, w] + d[w, v]
```

Let us consider comparisons where w = A:

	A	B	C	D
A	0	-1	∞	∞
B	∞	0	4	3
C	∞	∞	0	-2
D	2	∞	∞	0



Floyd-Warshall Algorithm

```
 8   foreach (Vertex w : G):
 9     foreach (Vertex u : G):
10       foreach (Vertex v : G):
11         if (d[u, v] > d[u, w] + d[w, v])
12           d[u, v] = d[u, w] + d[w, v]
```

Let **w** be midpoint

Let **u** be start point

Let **v** be end point

Is our distance shorter now?

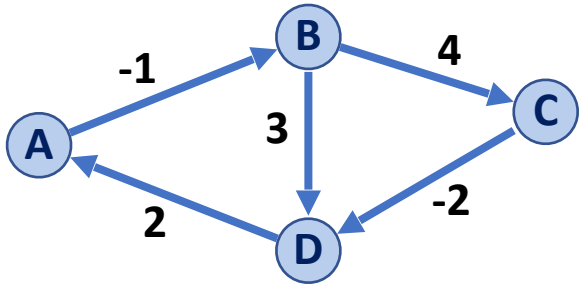
Let us consider comparisons where w = A:

u=A, v=A

u=A, v=B

Don't waste time if u=w or v=w!

	A	B	C	D
A	0	-1	∞	∞
B	∞	0	4	3
C	∞	∞	0	-2
D	2	∞	∞	0



Floyd-Warshall Algorithm

```
 8   foreach (Vertex w : G):
 9     foreach (Vertex u : G):
10       foreach (Vertex v : G):
11         if (d[u, v] > d[u, w] + d[w, v])
12           d[u, v] = d[u, w] + d[w, v]
```

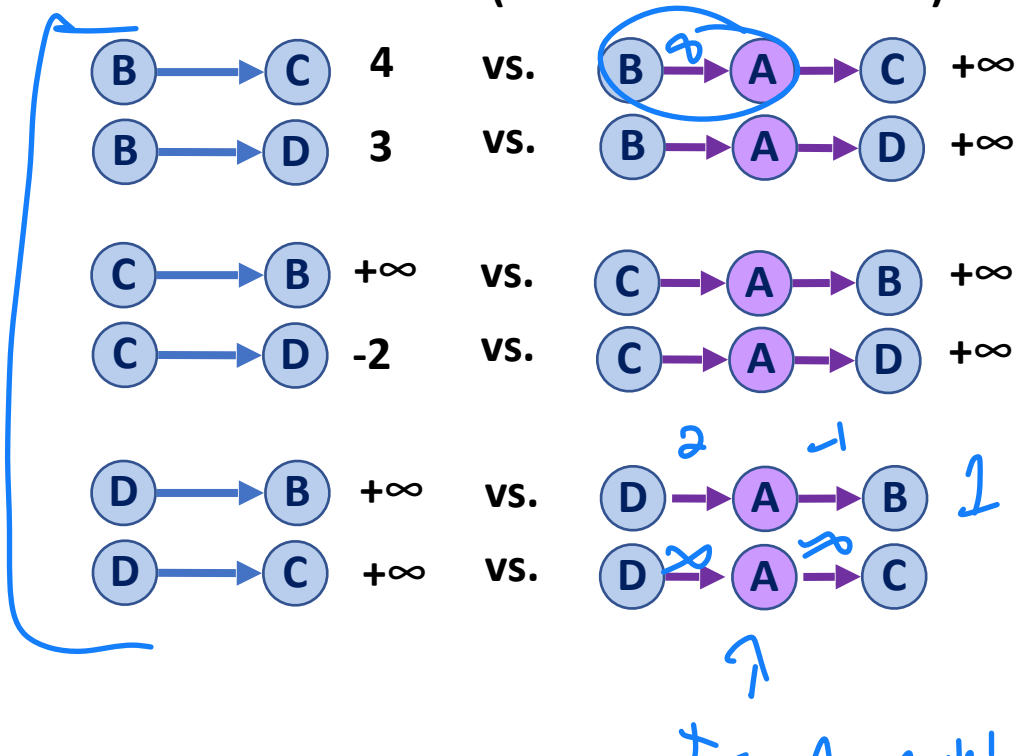
Let **w** be midpoint

Let **u** be start point

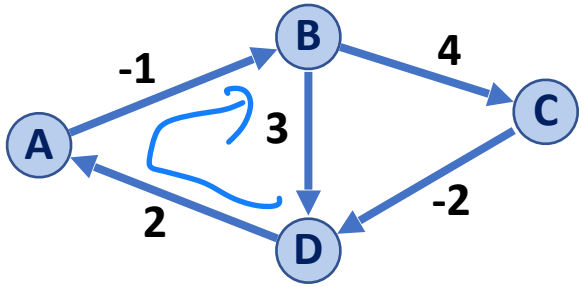
Let **v** be end point

Is our distance shorter now?

Let us consider w = A (and u != w and v != w):



	A	B	C	D
A	0	-1	∞	∞
B	∞	0	4	3
C	∞	∞	0	-2
D	2	∞	∞	0



Floyd-Warshall Algorithm

```
 8   foreach (Vertex w : G):
 9     foreach (Vertex u : G):
10       foreach (Vertex v : G):
11         if (d[u, v] > d[u, w] + d[w, v])
12           d[u, v] = d[u, w] + d[w, v]
```

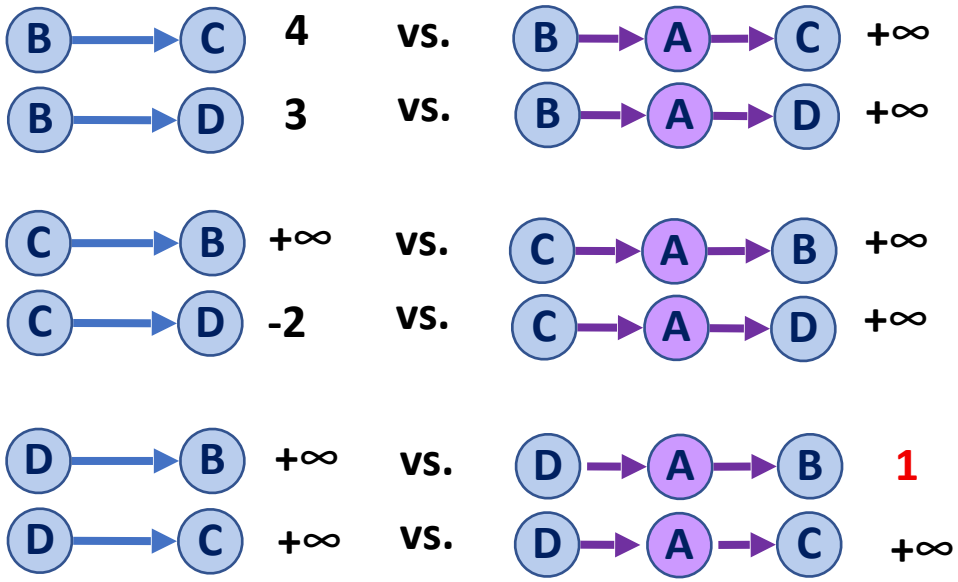
Let **w** be midpoint

Let **u** be start point

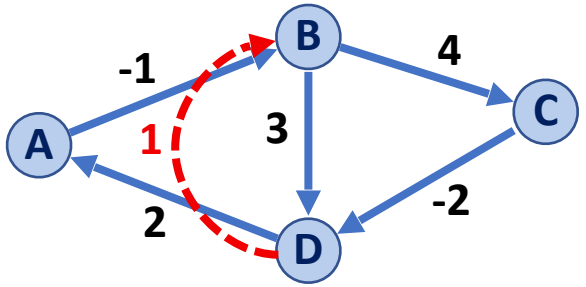
Let **v** be end point

Is our distance shorter now?

Let us consider w = A (and u != w and v != w):



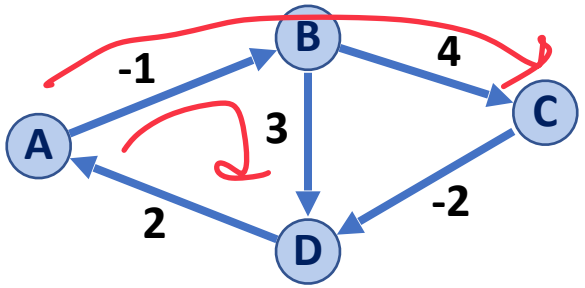
	A	B	C	D
A	0	-1	∞	∞
B	∞	0	4	3
C	∞	∞	0	-2
D	2	1	∞	0



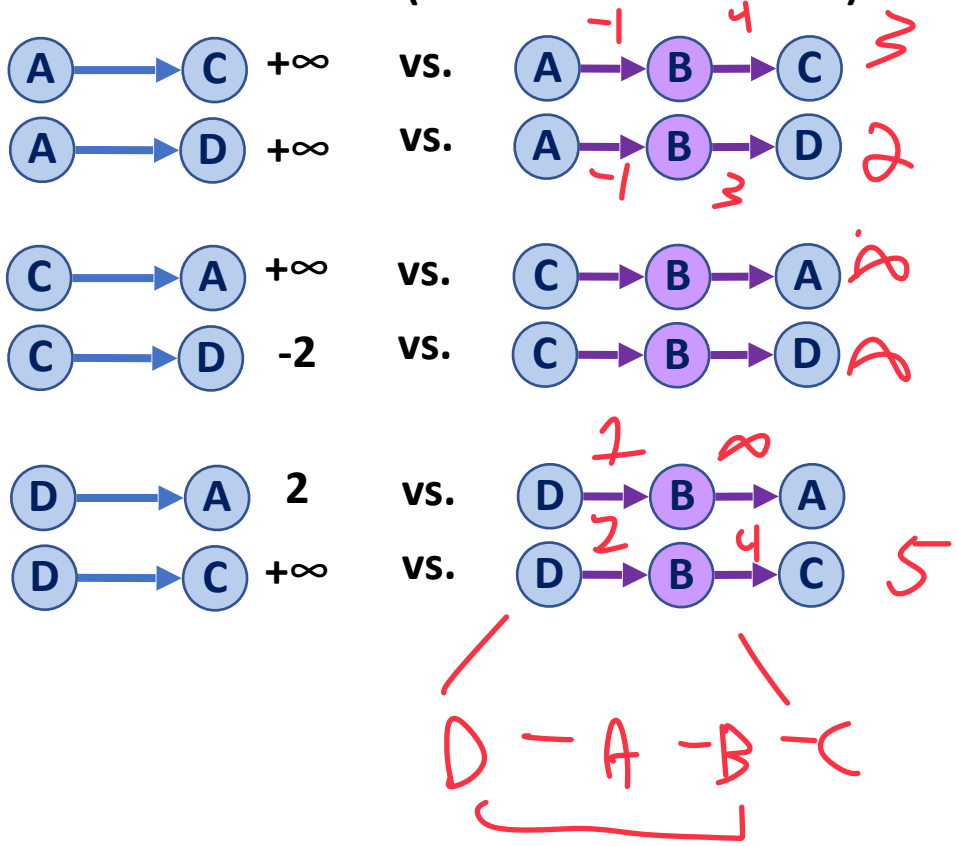
Floyd-Warshall Algorithm

```
 8   foreach (Vertex w : G):
 9     foreach (Vertex u : G):
10       foreach (Vertex v : G):
11         if (d[u, v] > d[u, w] + d[w, v])
12           d[u, v] = d[u, w] + d[w, v]
```

	A	B	C	D
A	0	-1	∞	∞
B	∞	0	4	3
C	∞	∞	0	-2
D	2	1	∞	0



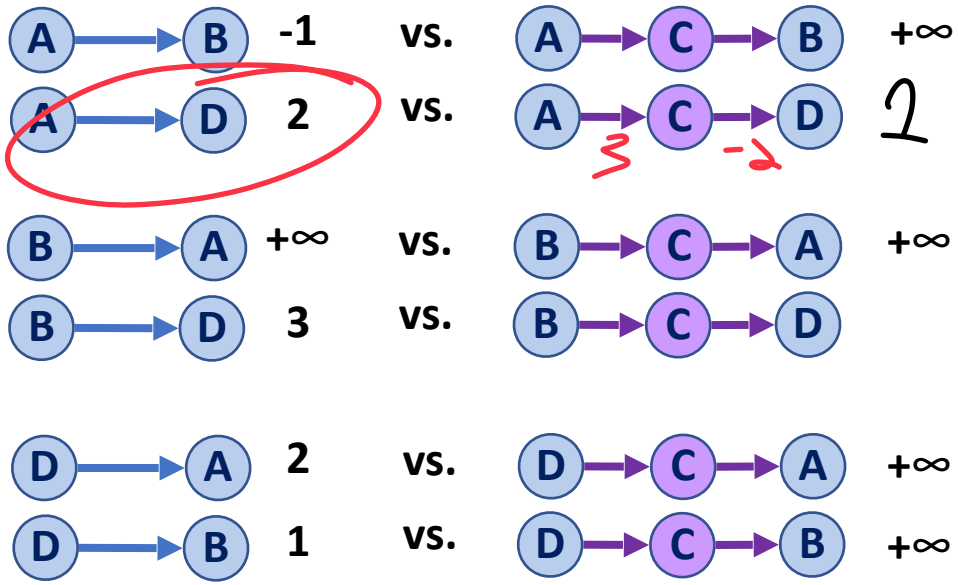
Let us consider w = B (and u != w and v != w):



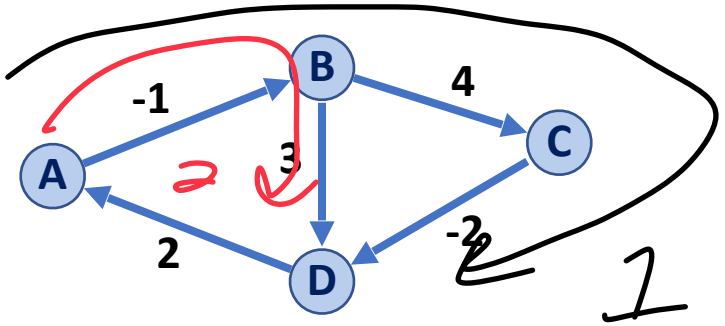
Floyd-Warshall Algorithm

```
 8   foreach (Vertex w : G):
 9     foreach (Vertex u : G):
10       foreach (Vertex v : G):
11         if (d[u, v] > d[u, w] + d[w, v])
12           d[u, v] = d[u, w] + d[w, v]
```

Let us consider w = C (and u != w and v != w):



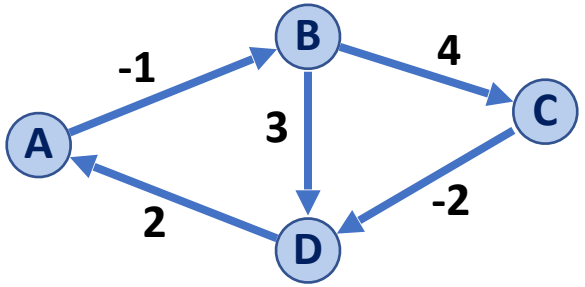
	A	B	C	D
A	0	-1	3	2
B	∞	0	4	3
C	∞	∞	0	-2
D	2	1	5	0



Floyd-Warshall Algorithm

```
 1  FloydWarshall(G):
 2    Let d be a adj. matrix initialized to +inf
 3    foreach (Vertex v : G):
 4      d[v][v] = 0
 5    foreach (Edge (u, v) : G):
 6      d[u][v] = cost(u, v)
 7
 8    foreach (Vertex u : G):
 9      foreach (Vertex v : G):
10        foreach (Vertex w : G):
11          if (d[u, v] > d[u, w] + d[w, v])
12            d[u, v] = d[u, w] + d[w, v]
```

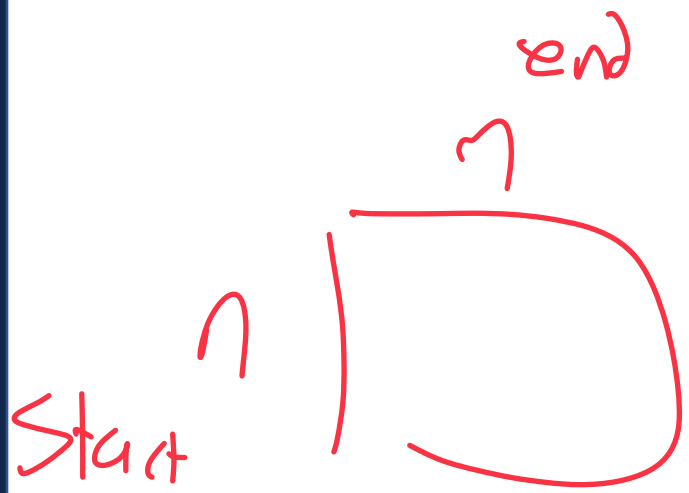
	A	B	C	D
A	0	-1	3	1
B	5	0	4	2
C	0	-1	0	-2
D	2	1	5	0



Floyd-Warshall Algorithm

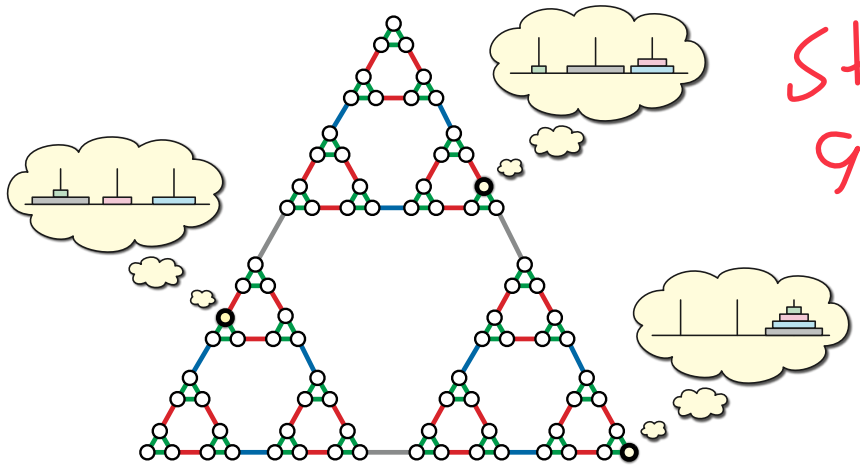
Running time? $O(n^3)$

↳ To produce shortest path from every start to every end




```
    FloydWarshall(G):
 6    Let d be a adj. matrix initialized to +inf
 7    foreach (Vertex v : G):
 8      d[v][v] = 0
 9    foreach (Edge (u, v) : G):
10      d[u][v] = cost(u, v)
11
12    foreach (Vertex u : G):
13      foreach (Vertex v : G):
14        foreach (Vertex w : G):
15          if d[u, v] > d[u, w] + d[w, v]:
16            d[u, v] = d[u, w] + d[w, v]
```

We have only scratched the surface on graphs!



State graph

↑

mp-puzzle

Image from Jeff Erickson Algorithms First Edition

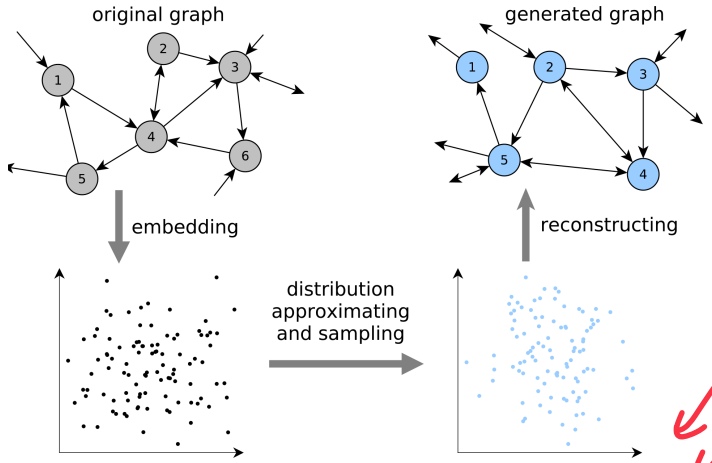
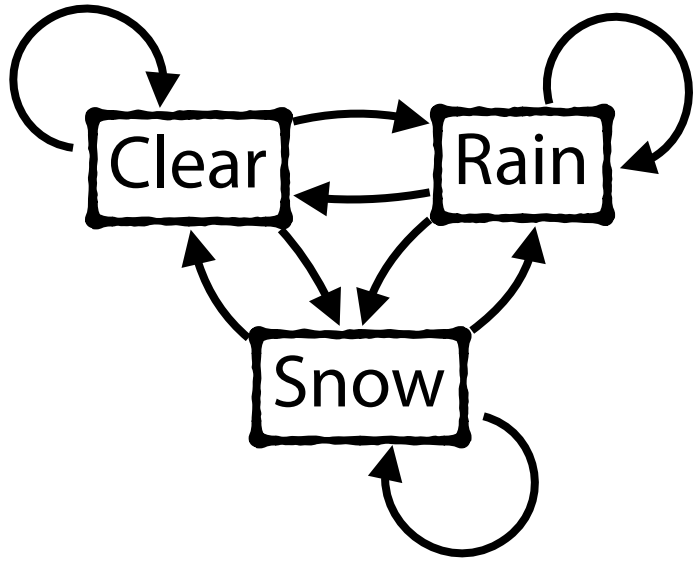


Image from Drobyshevskiy et al. **Random graph modeling: A survey of the concepts.** 2019



$$M = \begin{pmatrix} .5 & .3 & .2 \\ .5 & .4 & .1 \\ .2 & .1 & .7 \end{pmatrix}$$

Markov chain (Honors see this in 2-3 weeks

Randomized Algorithms

A **randomized algorithm** is one which uses a source of randomness somewhere in its implementation.

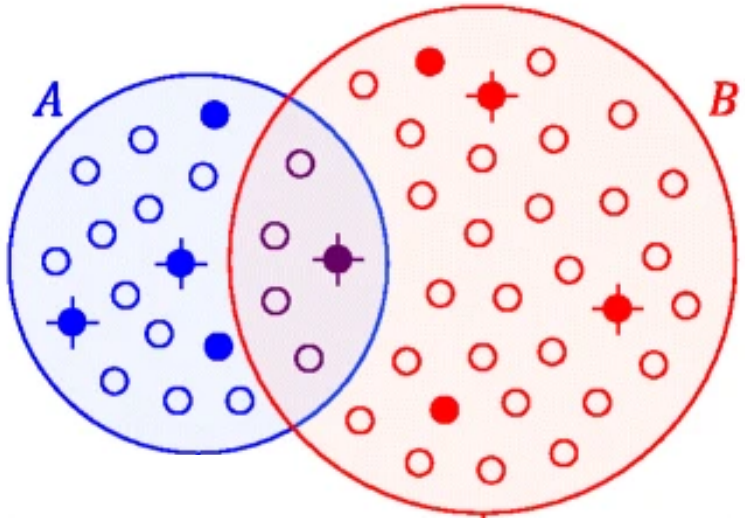
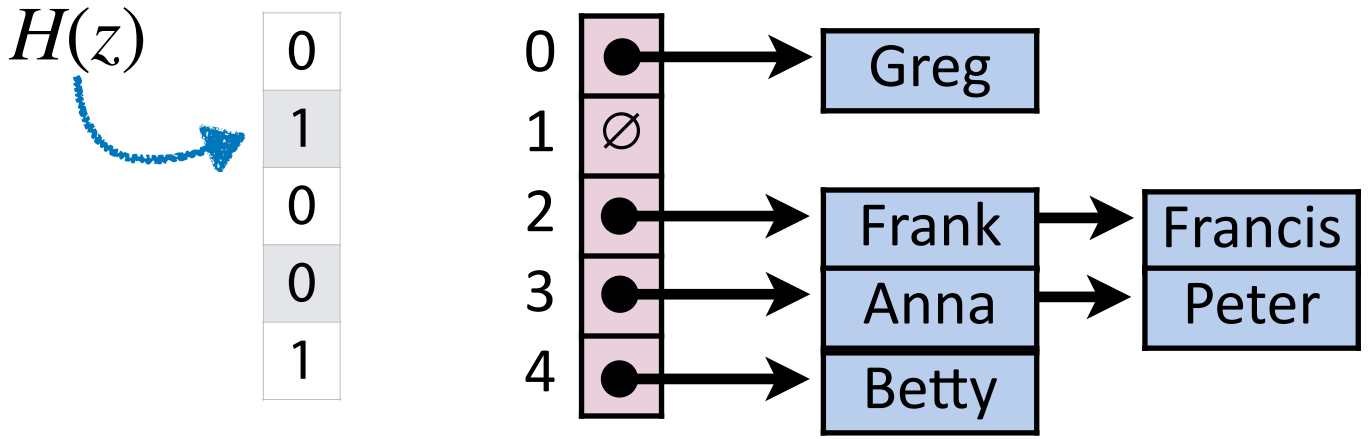


Figure from Ondov et al 2016



$H(x)$	0	2	1	0	0	4	0	2	0	6
$H(y)$	1	0	2	3	1	0	3	4	0	1
$H(z)$	2	1	0	2	0	1	0	0	7	2

A faulty list

Imagine you have a list ADT implementation ***except***…

Every time you called **insert**, it would fail 50% of the time.

Quick Primes with Fermat's Primality Test

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 \pmod{n}$

Probabilistic Accuracy: Fermat primality test

	$a^{p-1} \equiv 1 \pmod{p}$	$a^{p-1} \not\equiv 1 \pmod{p}$
p is prime		
p is not prime		

Probabilistic Accuracy: Fermat primality test

Let's assume $\alpha = .5$

First trial: $a = a_0$ and prime test returns 'prime!'

Second trial: $a = a_1$ and prime test returns 'prime!'

Third trial: $a = a_2$ and prime test returns 'not prime!'

Is our number prime?

What is our **false positive** probability? Our **false negative** probability?

Probabilistic Accuracy: Fermat primality test

Summary: Randomized algorithms can also have fixed (or bounded) runtimes at the cost of probabilistic accuracy.

Randomness:

Assumptions: