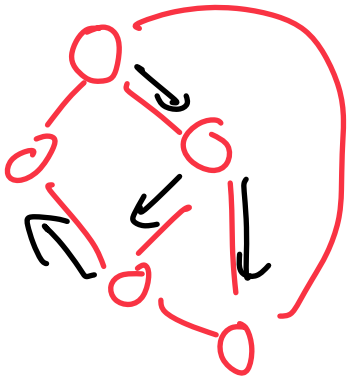
Data Structures

MST 2

CS 225
Brad Solomon

April 8, 2026



Department of Computer Science

Engineering Open House! On Friday

Learning Objectives

Review the minimum spanning tree (with weights)

Review Kruskal's / Prim's MST Algorithms

Focus on determining Big O of complex pseudocode

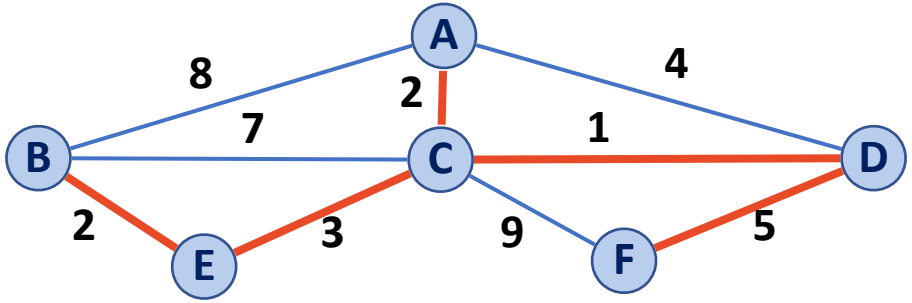
Compare implementations under different conditions

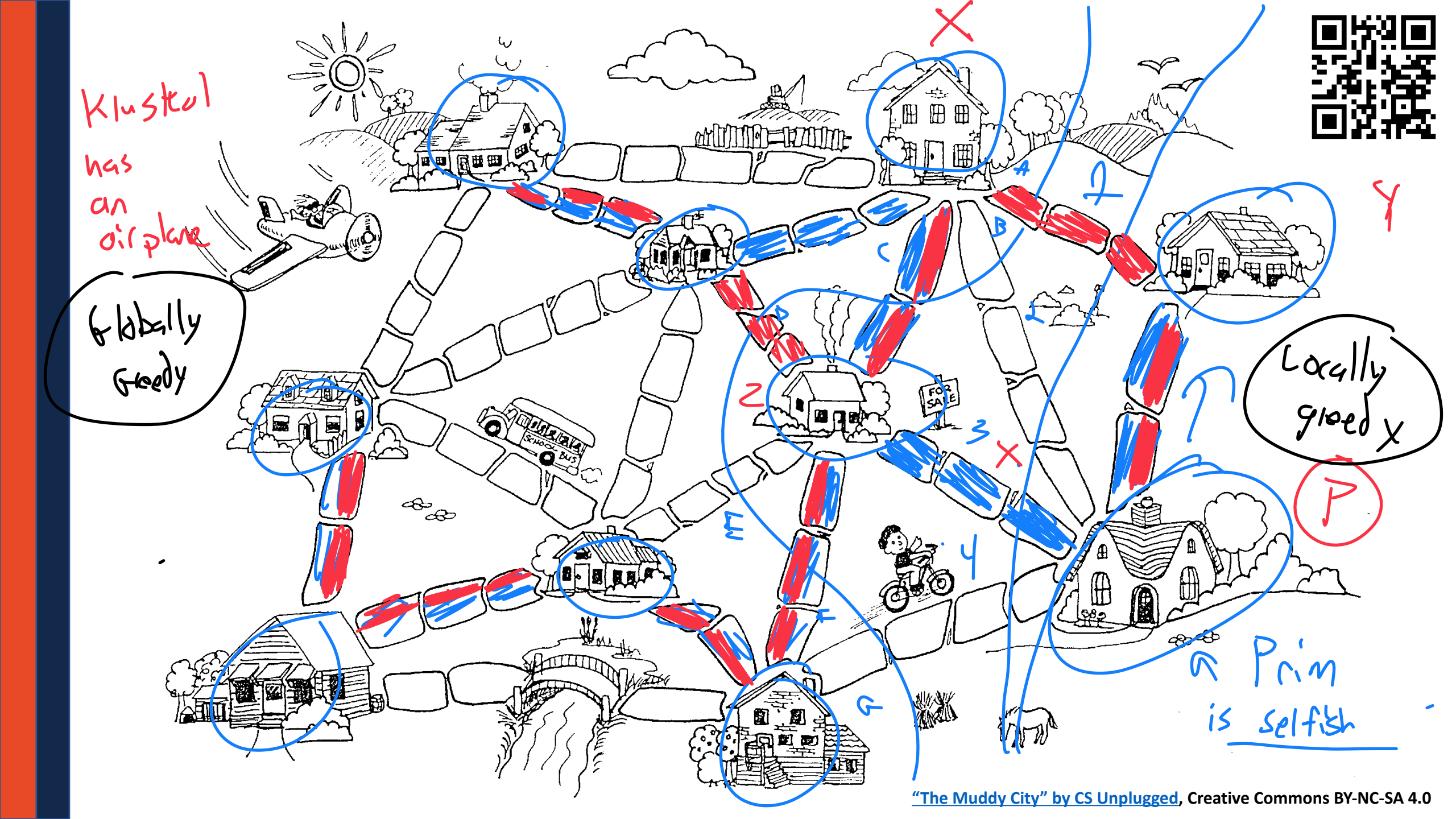
Minimum Spanning Tree Algorithms

Input: Connected, undirected graph **G** with edge weights (unconstrained, but must be additive)

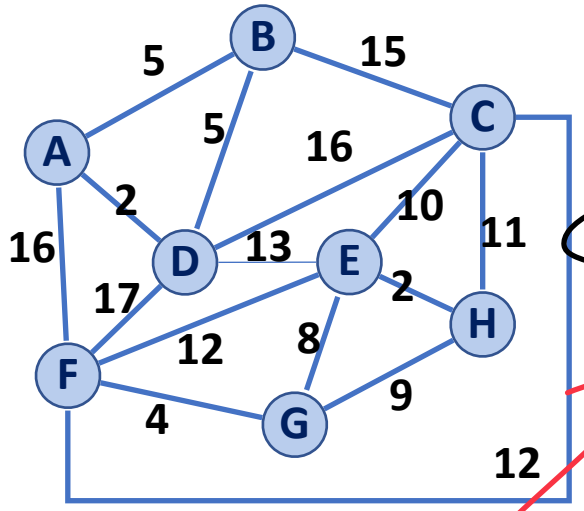
Output: A graph G' with the following properties:

- G' is a spanning graph of G → A connected acyclic tree containing all vertices in G
  ↳ n-1 edges
- G' is a tree (connected, acyclic)
- G' has a minimal total weight among all spanning trees





Kruskal's Algorithm *(A graph algorithm for the MST problem)*



All adjacent

What information do I need to get efficiently?

1) The global minimum edge weight 10%

2) Edge connections for a given vertex 70%

3) If two vertices are already connected 30%

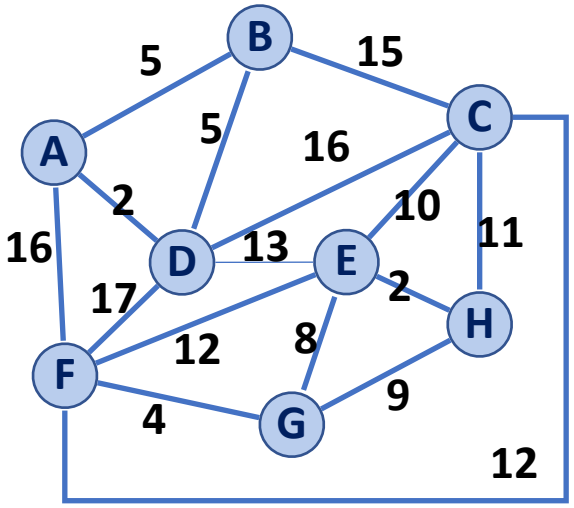
getEdge(v_1, v_8) 4) A visited list of vertices 60%

↓ or rather does it exist

↳ Not needed

Stop after n-1 edges

Kruskal's Algorithm *(A graph algorithm for the MST problem)*



What information do I need to get efficiently?

1) The global minimum edge weight

3) If two vertices are already connected

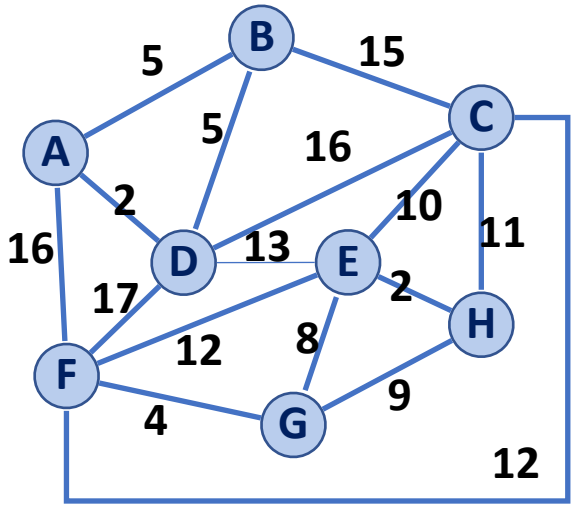
What does this tell me about my implementation choices?

1) We need a **priority queue** of edges (sorted by weight)

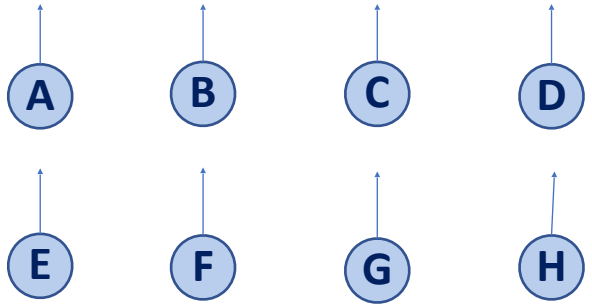
2) We need a **disjoint set** of vertices

Kruskal's Algorithm

(A, D)
(E, H)
(F, G)
(A, B)
(B, D)
(G, E)
(G, H)
(E, C)
(C, H)
(E, F)
(F, C)
(D, E)
(B, C)
(C, D)
(A, F)
(D, F)



M



1) Build a **priority queue** on edges

↳ Min heap

Or. . .

↳ Sorted array

2) Build a **disjoint set** on vertices

↳ Let every vertex start as own set

Repeatedly check if next smallest edge is in same set

↳ If not, record edge

union sets

why not stop? This connects edges to n vertices

Kruskal's Algorithm

(A, D)
(E, H)
(F, G)
(A, B)
(B, D)
(G, E)
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(E, C)
(C, H)
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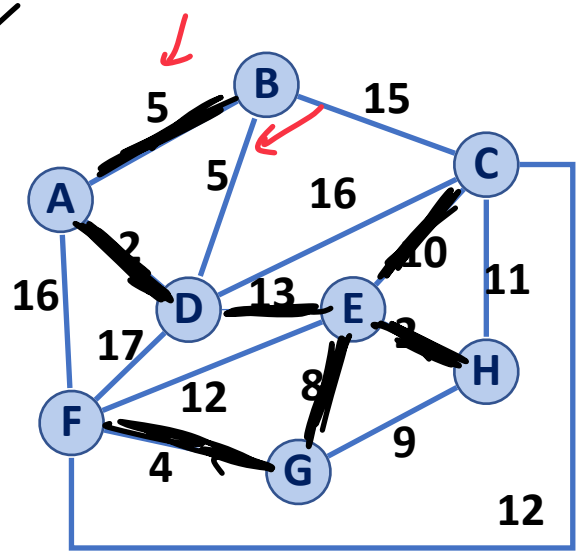
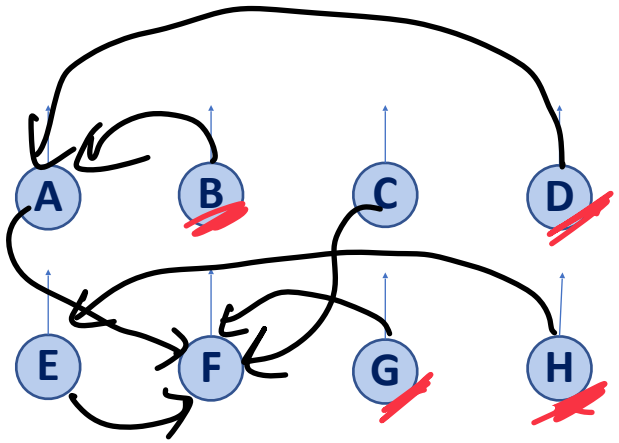
(D, F)





1) Build a **priority queue** on edges

A minheap

or

A sorted array

2) Build a **disjoint set** on vertices

All vertices start as their own set

3) Loop through min edges

If edge connects two disjoint sets

Union sets and record edge in MST

4) Stop when:

N-1 edges recorded

Same end condition

Only a single disjoint set remains

Kruskal's Algorithm

```
1  KruskalMST(G):
2    DisjointSets forest
3    foreach (Vertex v : G.vertices()):
4      forest.makeSet(v)
5
6    PriorityQueue Q     // min edge weight
7    Q.buildFromGraph(G.edges())
8
9    Graph T = (V, {})
10
11   while |T.edges()| < n-1:
12     Vertex (u, v) = Q.removeMin()
13     if forest.find(u) != forest.find(v):
14        T.addEdge(u, v)
15        forest.union( forest.find(u),
16                      forest.find(v) )
17
18   return T
19
```

(2)

(1)

(3)

1) Build a **priority queue** on edges

A minheap

or

A sorted array

2) Build a **disjoint set** on vertices

All vertices start as their own set

3) Loop through min edges

If edge connects two disjoint sets

Union sets and record edge in MST

4) Stop when:

N-1 edges recorded

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Kruskal's Algorithm

|V| = n, |E| = m

What is the Big O?

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```

O(n)

assume min heap (for now)

O(m)

we might visit all edges to find edges in MST

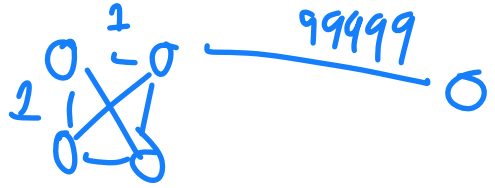
mx

Mx

O(1)

O(1)

log(m)



20 more seconds

Kruskal's Algorithm

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$|V| = n, |E| = m$

What is the Big O?

2 — 4: O(n)

6 — 7: Heap: O(m)
Sorted List: O(m log m)

11: m x <12-17>

12—17: Heap: O(log m)
Sorted List: O(1)

$O(1 + n + m \log m)$

Disjoint set we treat as O(1) b/c path compression w/ smart union

Kruskal's Algorithm

Priority Queue:	Heap	Sorted Array
Building :7	$O(m)$	$O(m \log m)$
Each removeMin :12	$O(\log m)$	$O(1)$

$O(n + m \log m)$ overall

```
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Why heap?

↳ Worst case line 11 visits m edges
In practice is that common?

↳ heap shrinks over time ←

↳ what if edge weights change? (Dynamic graph)

Why sorted array?

↳ The sorted array is not destroyed during Kruskal

Kruskal's Algorithm

Priority Queue:	Heap	Sorted Array
Building :7	O(m)	O(m log m)
Each removeMin :12	O(m log m)	O(m)

Both result in m + m log m

Why is heap good?

If edge weights can change!

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16                       forest.find(v) )
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```

Why is sorted array good?

Sorted array not destroyed and can be useful in other algorithms!

Kruskal's Algorithm

Priority Queue:	Total Running Time
Heap	$O(n + m \log m)$
Sorted Array	$O(n + m \log m)$

```
1  KruskalMST(G):
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```

Kruskal's Algorithm

O(log n) → O(log n^2) → O(2 log n) → O(log n)

Priority Queue:	Total Running Time
Heap	O(n)+O(m) + O(m log m)
Sorted Array	O(n) + O(m log m) + O(m)

simple connected graph

Neat trick: O(log m) = O(log n)

Bounds of n & m relationship?

$$n - 1 \leq m \leq n^2$$

Connected graph *Complete graph* $\frac{n(n-1)}{2}$

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```
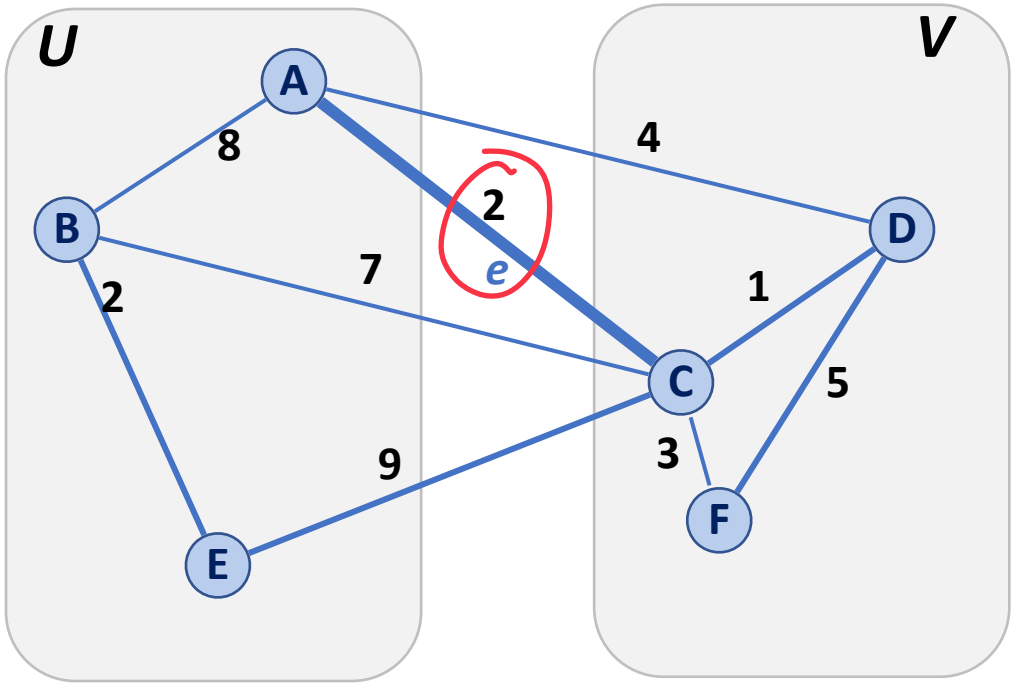

Partition Property

Consider an arbitrary partition of the vertices on **G** into two subsets **U** and **V**.

Let **e** be an edge of minimum weight across the partition.

Then **e** is part of some minimum spanning tree.



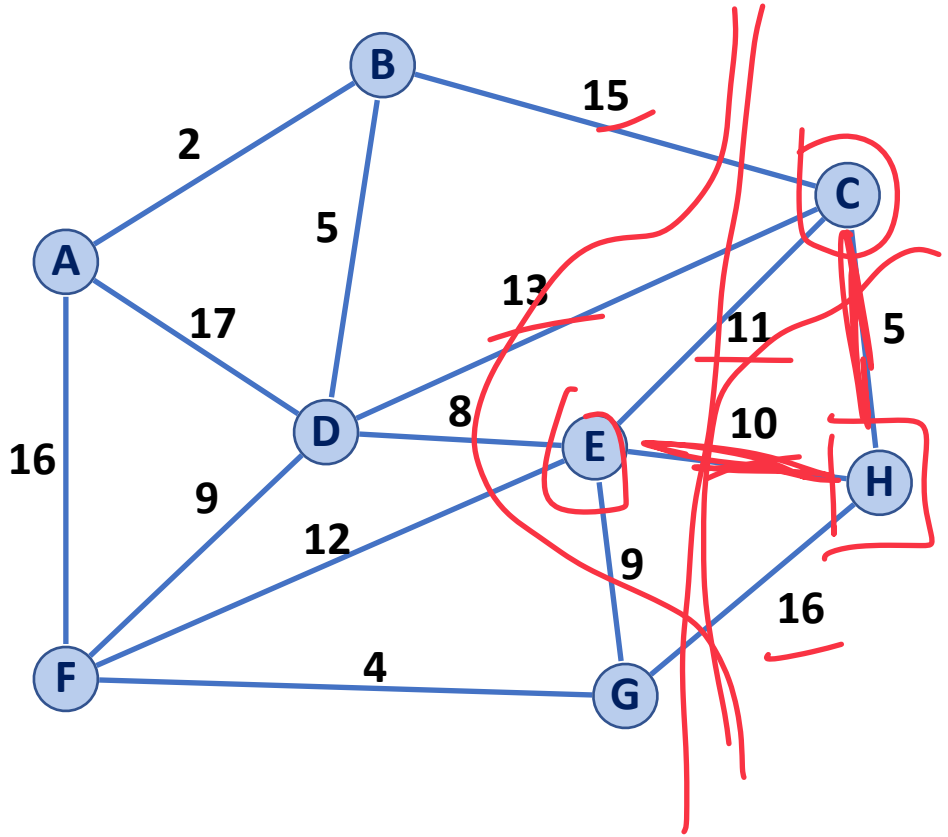
Out

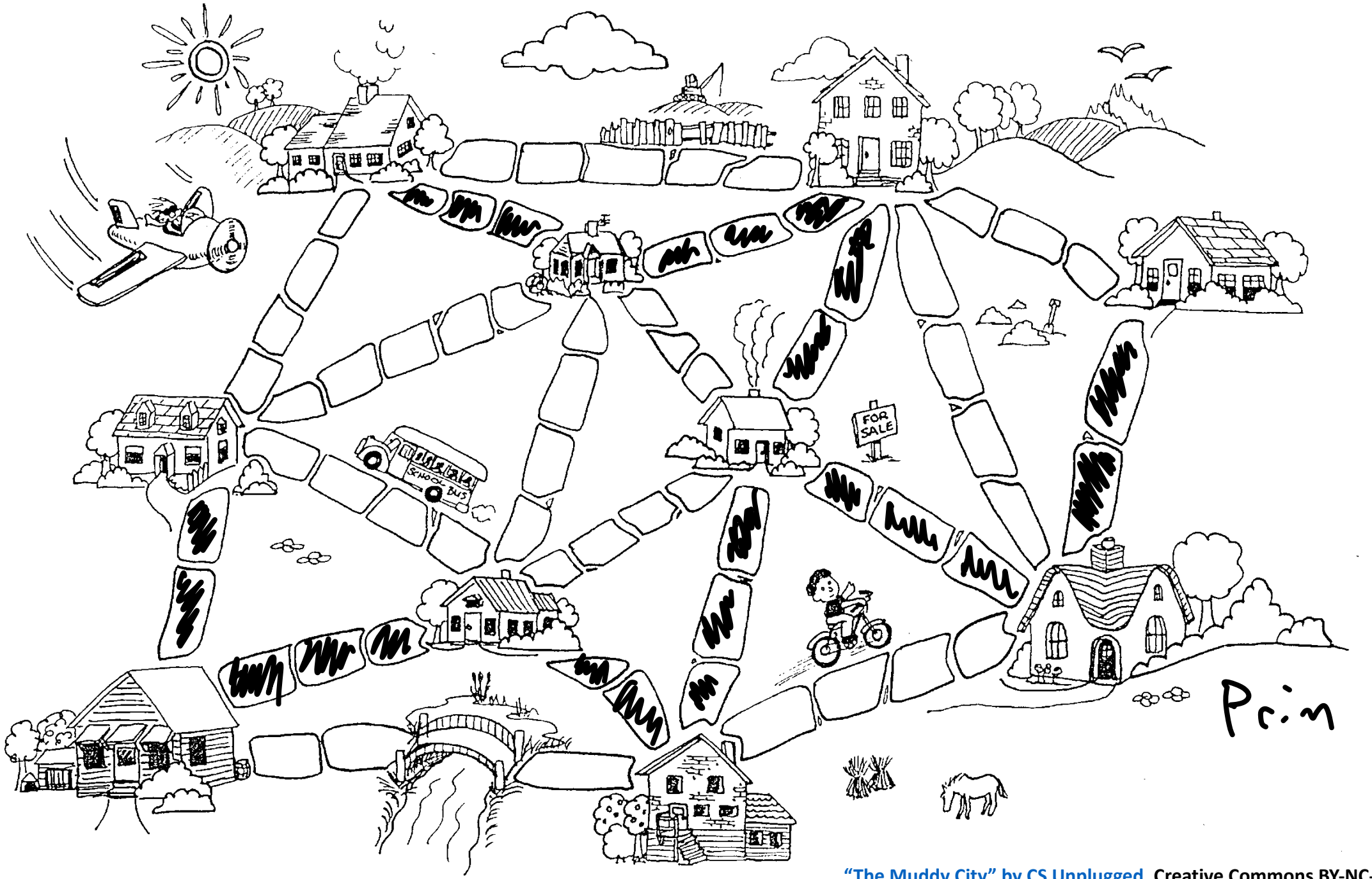
In

Locally Greedy

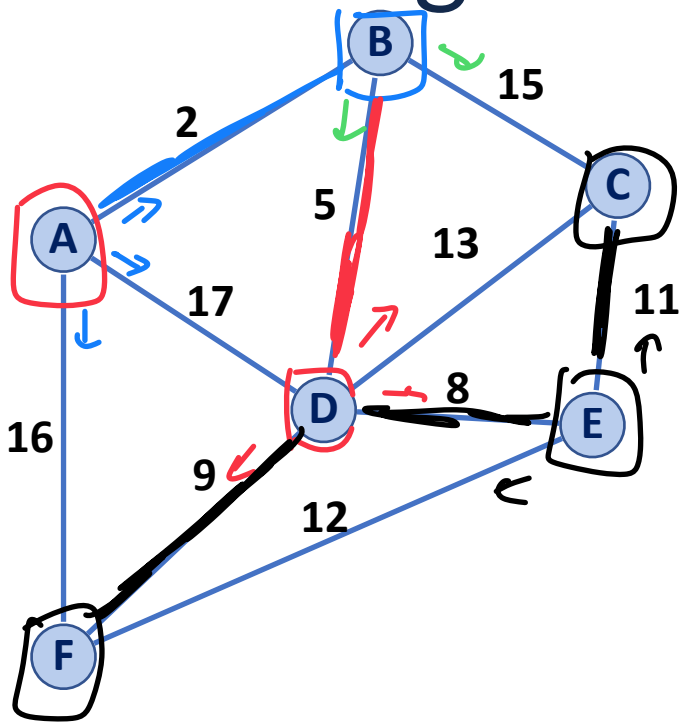
Partition Property

The partition property suggests an algorithm:





Prim's Algorithm



A	B	C	D	E	F
0, –	~~∞~~ 2, A	~~∞~~ ~~15, B~~ 13, A	~~∞~~ 17, A	~~∞~~ 8, D	~~∞~~ ~~16, A~~ 9, D

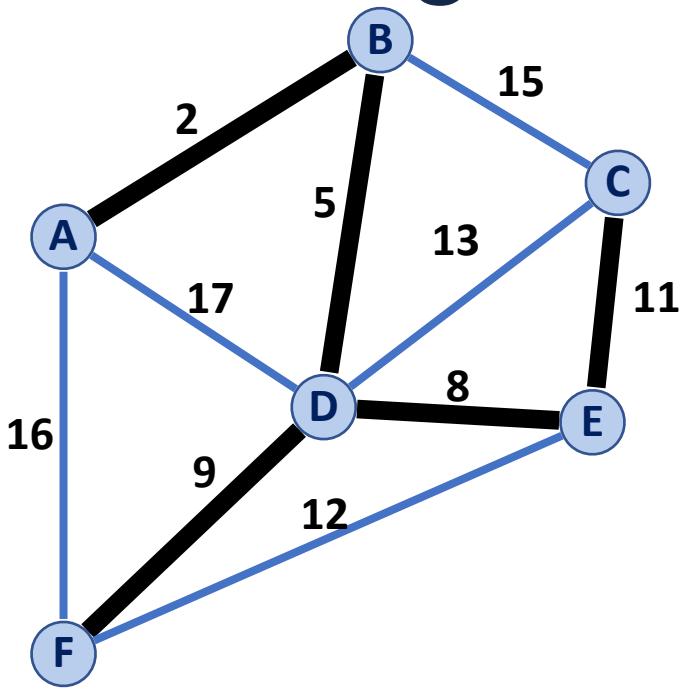
15, D 5, B 11, C

```
1  PrimMST(G, s):
2    Input: G, Graph;
3           s, vertex in G, starting vertex
4    Output: T, a minimum spanning tree (MST) of G
5
6    foreach (Vertex v : G.vertices()):
7      d[v] = +inf
8      p[v] = NULL
9    d[s] = 0
10
11   PriorityQueue Q   // min distance, defined by d[v]
12   Q.buildHeap(G.vertices())
13   Graph T           // "labeled set"
14
15   repeat n times:
16     Vertex m = Q.removeMin()
17     T.add(m)
18     foreach (Vertex v : neighbors of m not in T):
19       if cost(v, m) < d[v]:
20         d[v] = cost(v, m)
21         p[v] = m
22
23   return T
```

Add the vertex to my MST

edge through m is smaller than current dist update edge

Prim's Algorithm



A	B	C	D	E	F
0, —	2, A	11, E	5, B	8, D	9, D

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Prim's Big O

$|V| = n, |E| = m$

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```

Prim's Big O

|V| = n, |E| = m

7 — 9: O(n)

Assign label to vertex

12—14:

MinHeap: O(n)

Unsorted Array: O(1)

16—22: Complicated!

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```
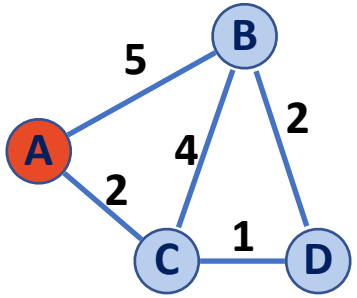
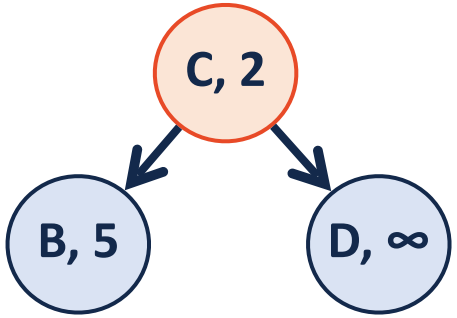

Depends on choice of **PriorityQueue** (MinHeap vs Unsorted Array)

Depends on choice of **Graph** (Adjacency Matrix vs Adjacency List)

A	B	C	D
0	5	2	∞



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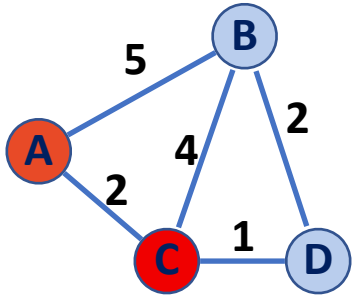
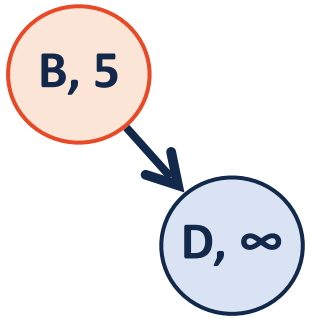
```

	Adj. Matrix	Adj. List
Heap	O(n) + _____ + O(n^2) + _____	O(n) + _____ + O(m) + _____

A	B	C	D
0	5	2, A	∞



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```
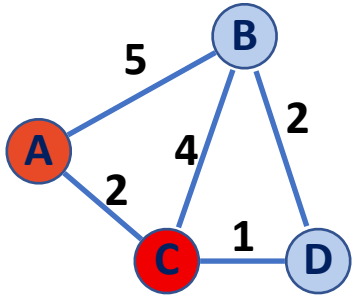
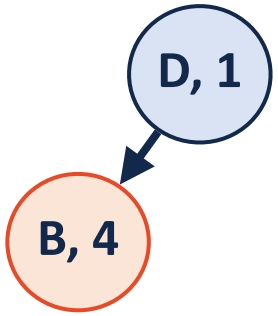

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + _____	O(n) + O(n log n) + O(m) + _____

A	B	C	D
0	4	2, A	1



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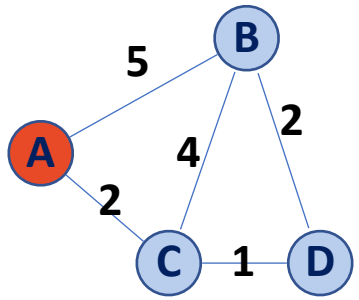
```

1) Change minheap value

2) HeapifyUp()

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + O(m log n)	O(n) + O(n log n) + O(m) + O(m log n)

(A, 0)
(D, ∞)
(C, 2)
(B, 5)

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17      Vertex m = Q.removeMin()
18      T.add(m)
19      foreach (Vertex v : neighbors of m not in T):
20        if cost(v, m) < d[v]:
21          d[v] = cost(v, m)
22          p[v] = m
23
```

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array		

Prim's Algorithm

Sparse Graph:

Dense Graph:

```
 6  PrimMST(G, s):
 7    foreach (Vertex v : G.vertices()):
 8      d[v] = +inf
 9      p[v] = NULL
10    d[s] = 0
11
12    PriorityQueue Q // min distance, defined by d[v]
13    Q.buildHeap(G.vertices())
14    Graph T          // "labeled set"
15
16    repeat n times:
17      Vertex m = Q.removeMin()
18      T.add(m)
19      foreach (Vertex v : neighbors of m not in T):
20        if cost(v, m) < d[v]:
21          d[v] = cost(v, m)
22          p[v] = m
23
```

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array	$O(n^2)$	$O(n^2)$

MST Algorithm Runtime:

Kruskal's Algorithm:
O(n + m log (n))

Prim's Algorithm:
O(n log(n) + m log (n))

Sparse Graph:

Dense Graph:

Suppose I have a new heap:

	Binary Heap	Fibonacci Heap
Remove Min	O(lg(n))	O(lg(n))
Decrease Key	O(lg(n))	O(1)*

What's Prim's updated running time?

```
    PrimMST(G, s):
 6    foreach (Vertex v : G.vertices()):
 7      d[v] = +inf
 8      p[v] = NULL
 9    d[s] = 0
10
11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T          // "labeled set"
14
15    repeat n times:
16      Vertex m = Q.removeMin()
17      T.add(m)
18      foreach (Vertex v : neighbors of m not in T):
19        if cost(v, m) < d[v]:
20          d[v] = cost(v, m)
21          p[v] = m
```

Shortest Path