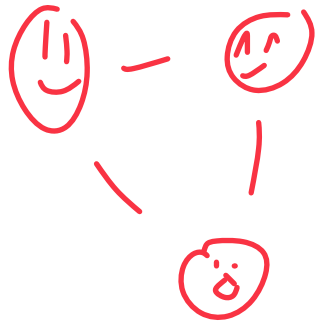
Data Structures

Graph Fundamentals

CS 225

Brad Solomon

March 27, 2026



UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Department of Computer Science

Learning Objectives

Define graph vocabulary

Discuss graph implementation and storage strategies

↳ 3 major graph implementations

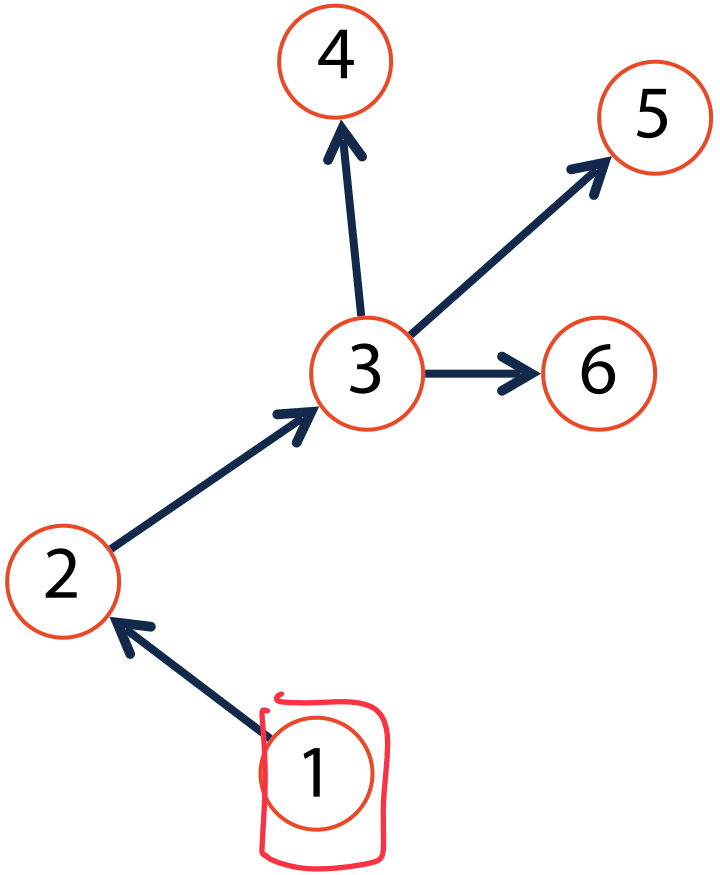
Whats next?

A non-linear data structure defined recursively as a collection of nodes where each node contains a value and zero or more connected nodes.

(In CS 225) a tree is also:

1) Acyclic — contains no cycles

2) Rooted — root node connected to all nodes



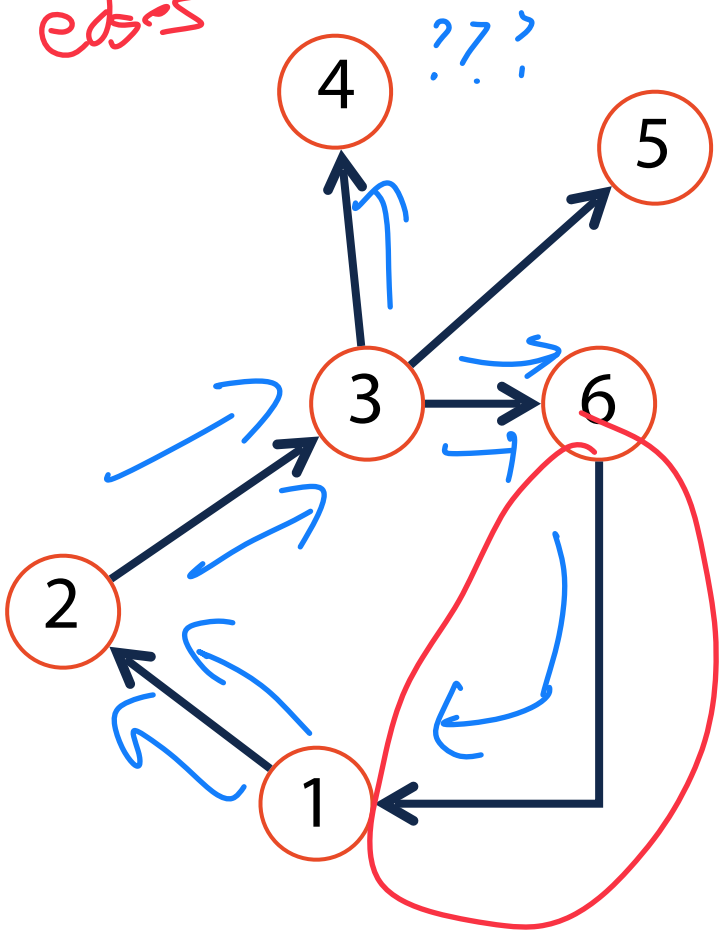
Whats next?

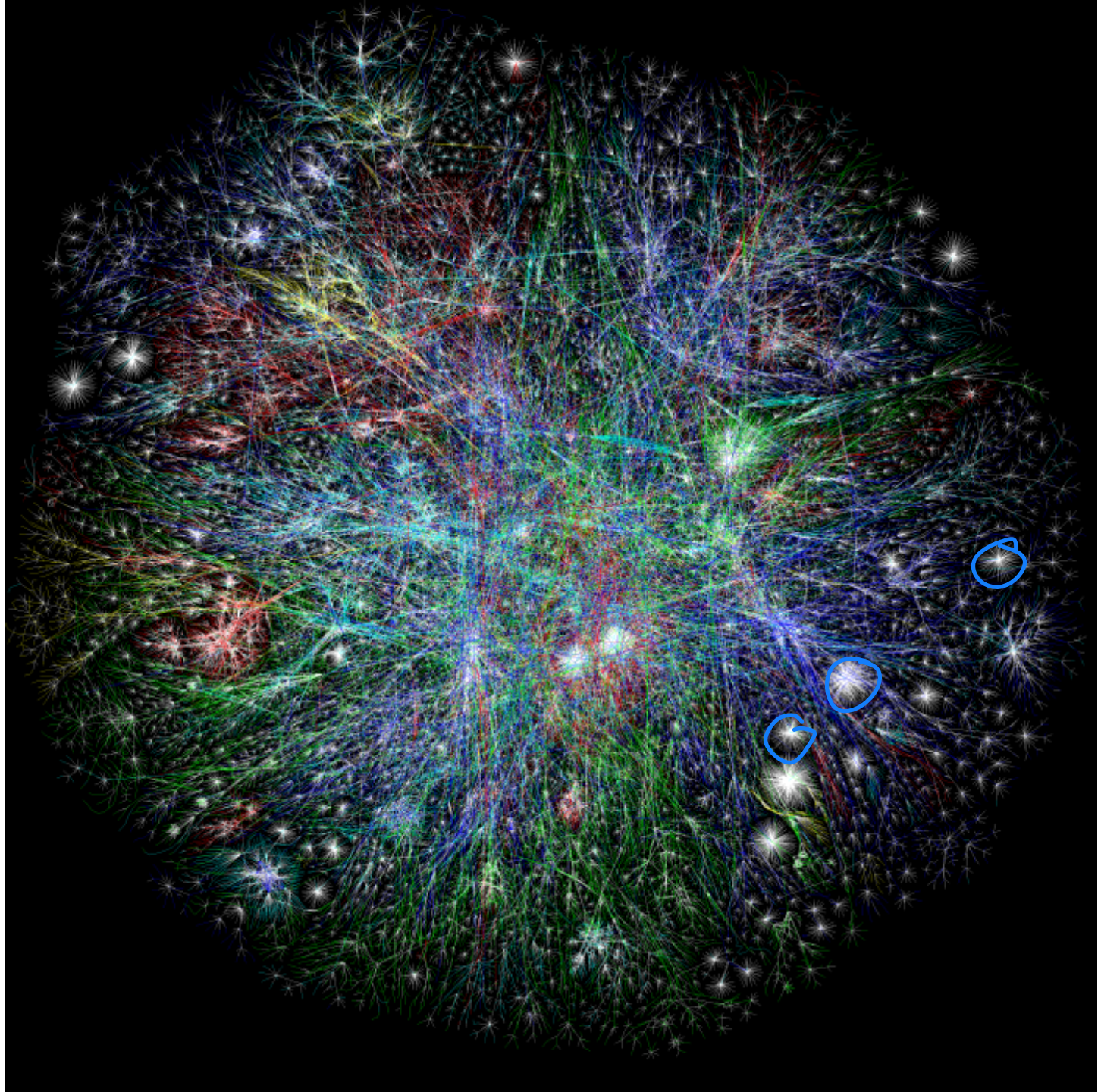
A non-linear data structure defined recursively as a collection of nodes where each node contains a value and zero or more connected nodes.

(In CS 225) a graph:

1) Can contain Cycles

2) Has no clear root





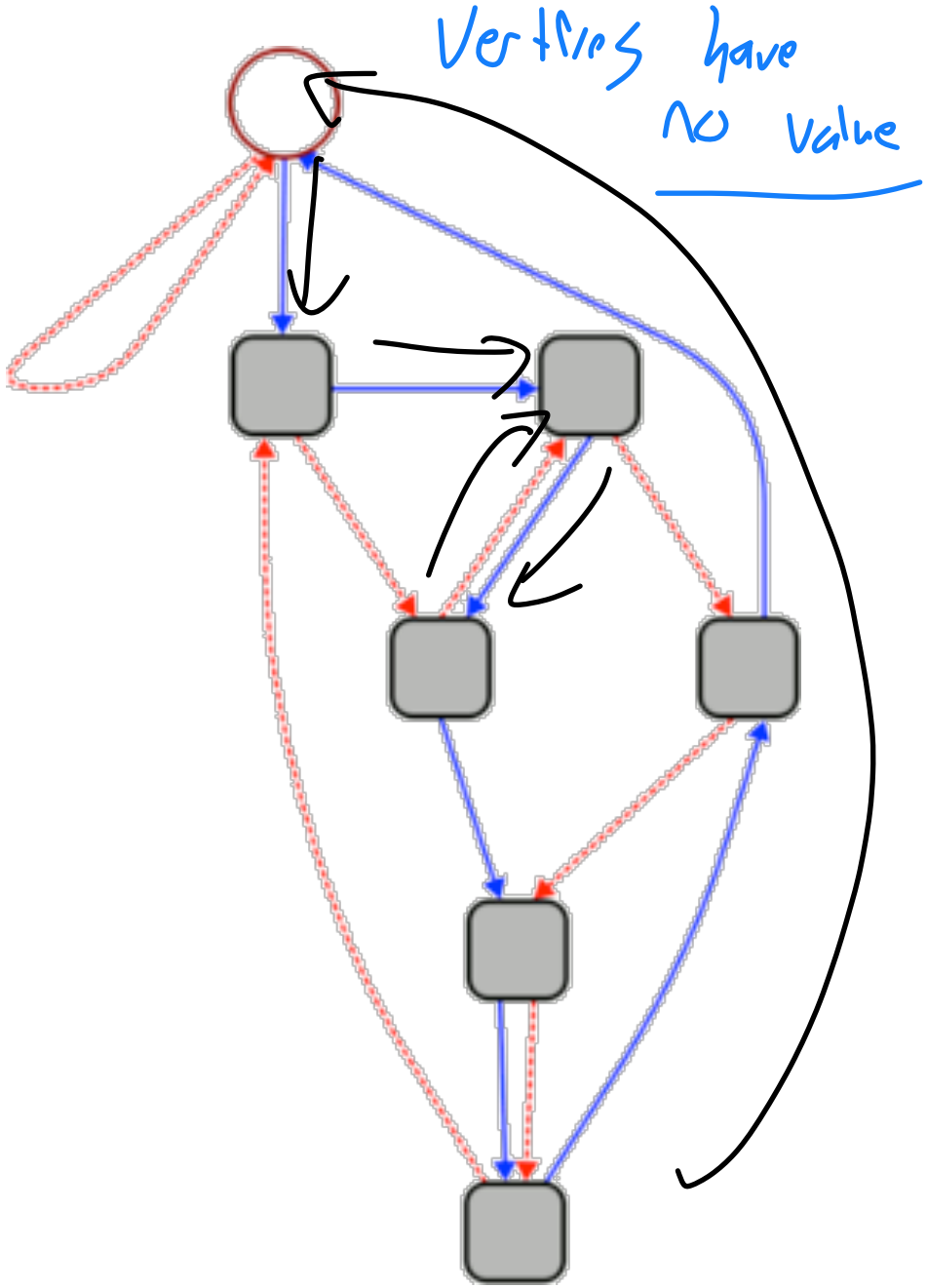
Nodes: Routers and servers

Edges: Connections

Big data graph

The Internet 2003

The OPTE Project (2003)



This graph can be used to quickly calculate whether a given number is divisible by 7.

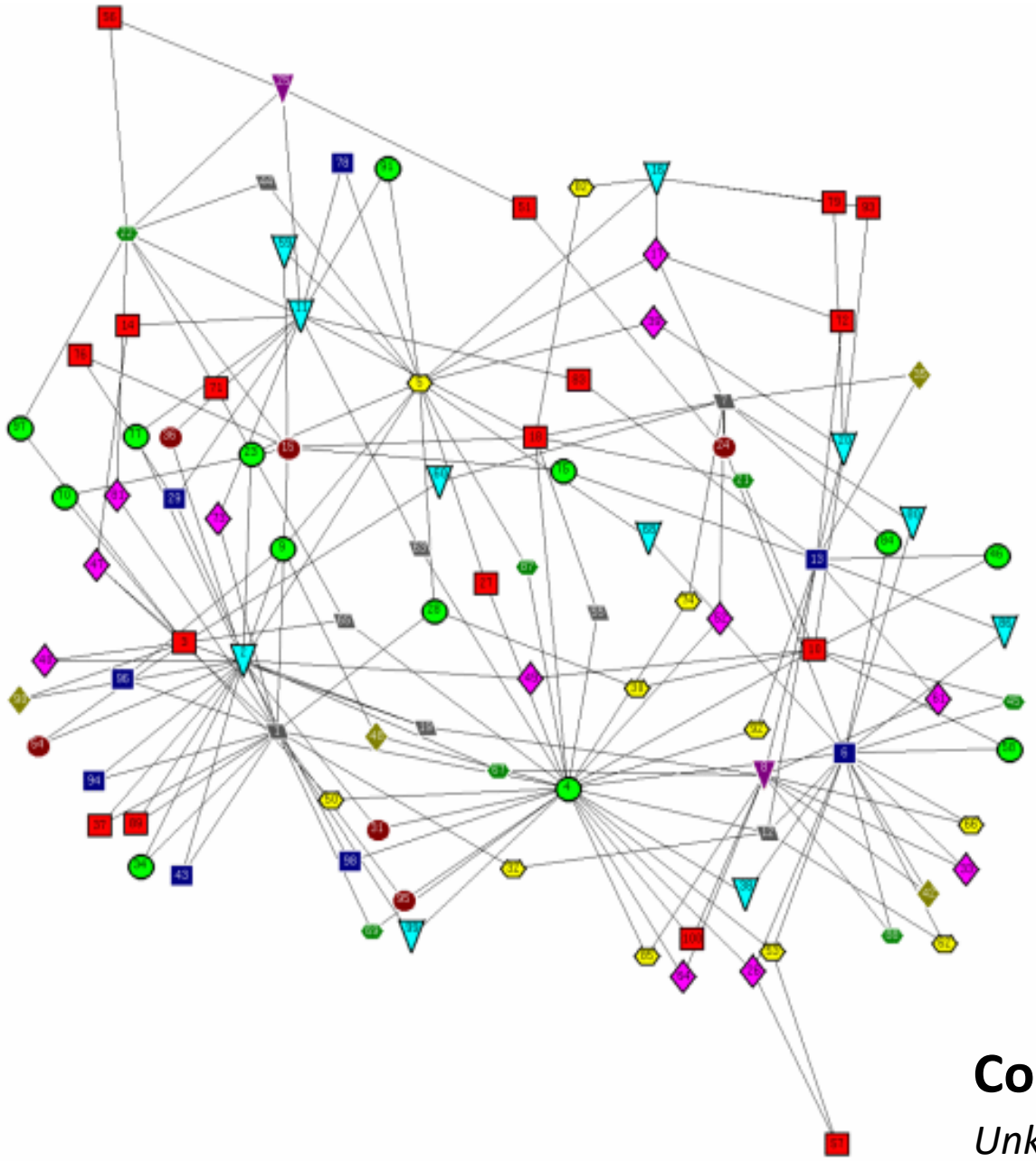
1. Start at the circle node at the top.
2. For each digit **d** in the given number, follow **d** blue (solid) edges in succession. As you move from one digit to the next, follow **1** red (dashed) edge.
3. If you end up back at the circle node, your number is divisible by 7.

3703 is divisible by 7

"Rule of 7"

Unknown Source

Presented by Cinda Heeren, 2016



Graph coloring

Vertices: classes

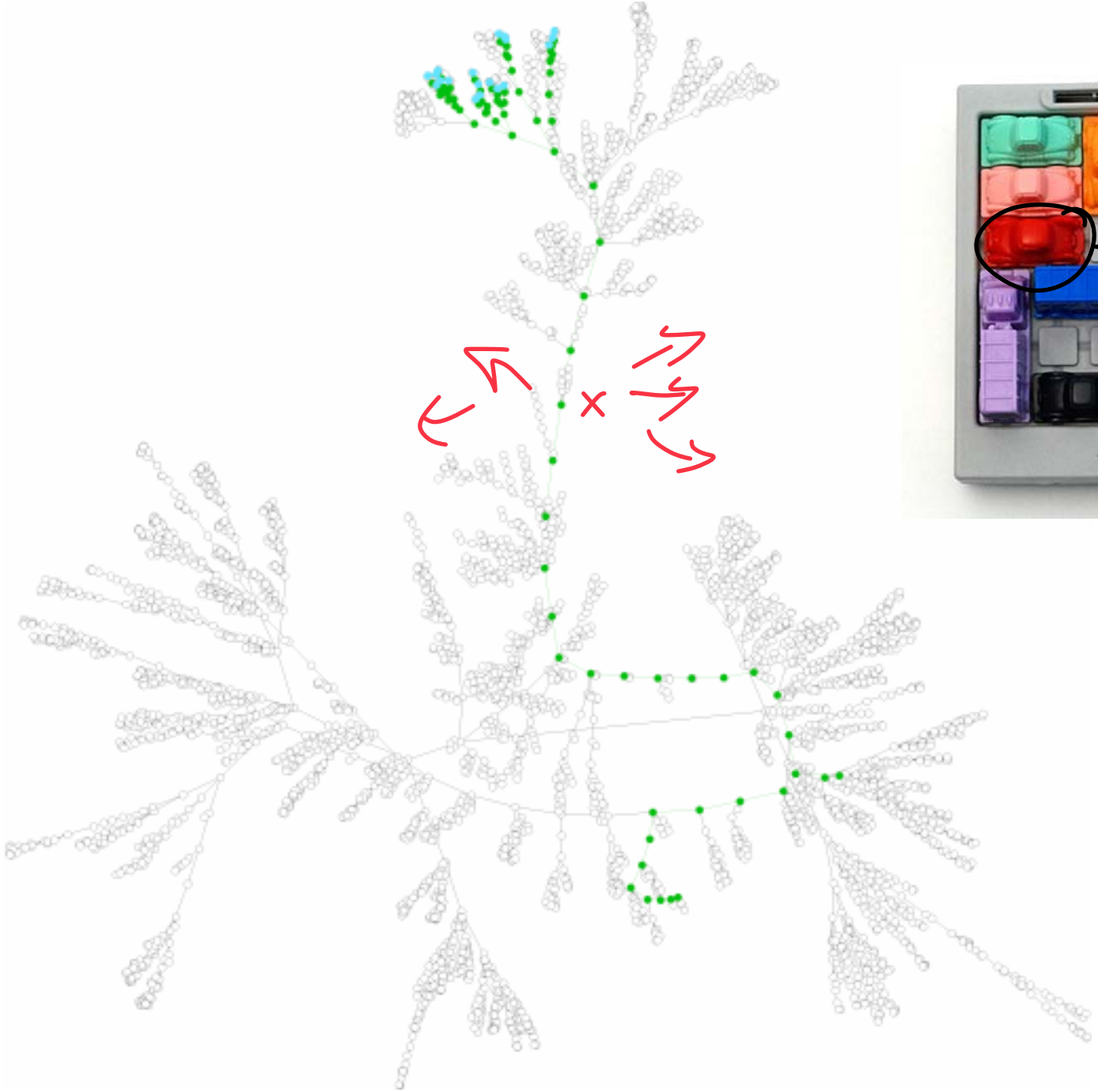
Edges: A student shares both classes

Color = time slot

Conflict-Free Final Exam Scheduling Graph

Unknown Source

Presented by Cinda Heeren, 2016



1 2 3 4
5 6 7 8

MP Puzzle
↳ Extremly large!

“Rush Hour” Solution

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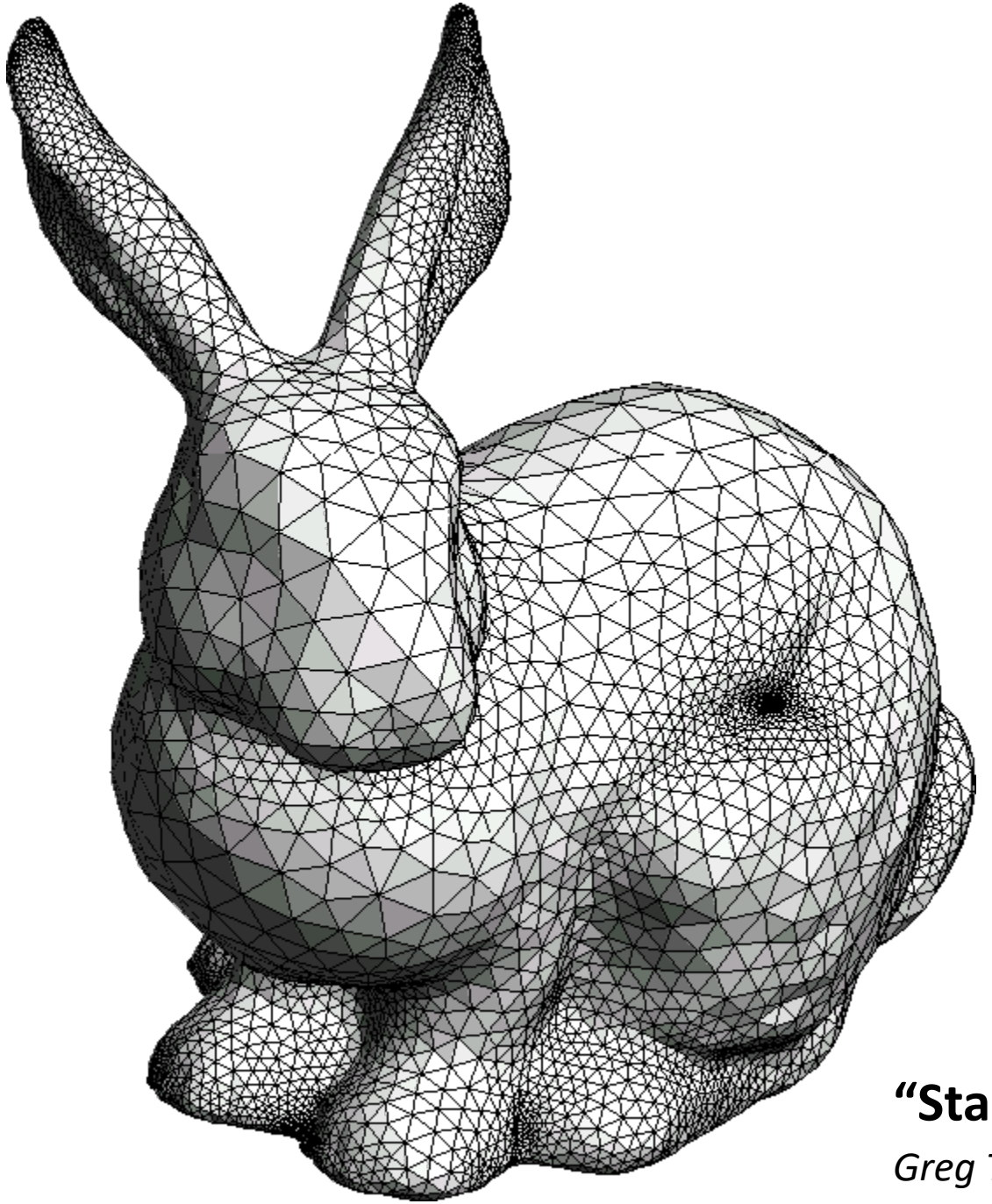
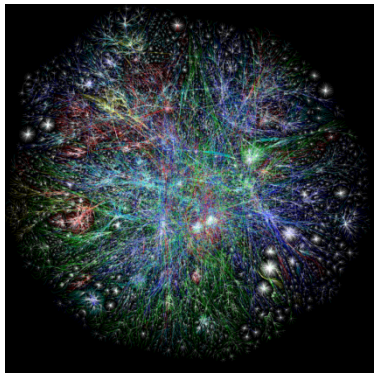


Graphics

“Stanford Bunny”

Greg Turk and Mark Levoy (1994)

Graphs



today

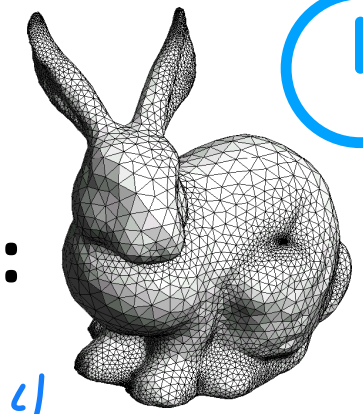
today next lecture

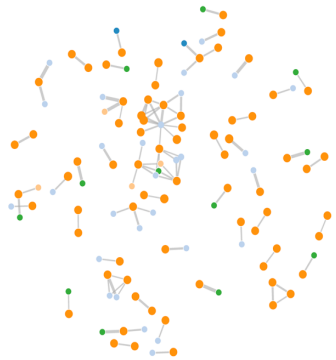
To study all of these structures:

1. A common vocabulary
2. Graph implementations
3. Graph traversals
4. Graph algorithms

exam 4

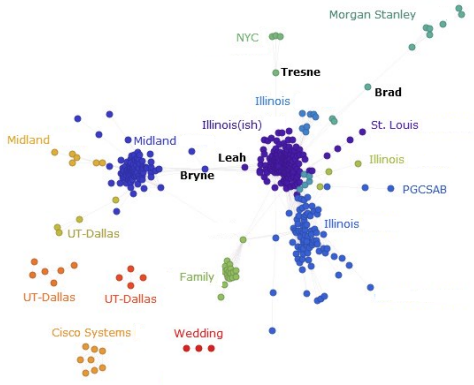
exam 5

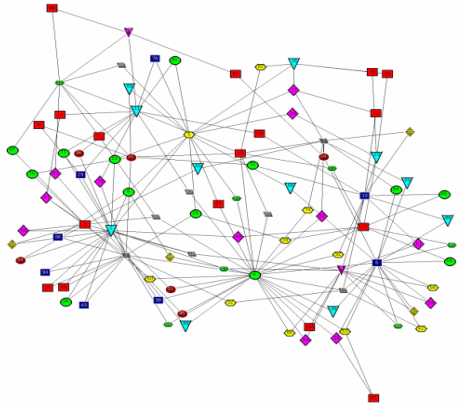


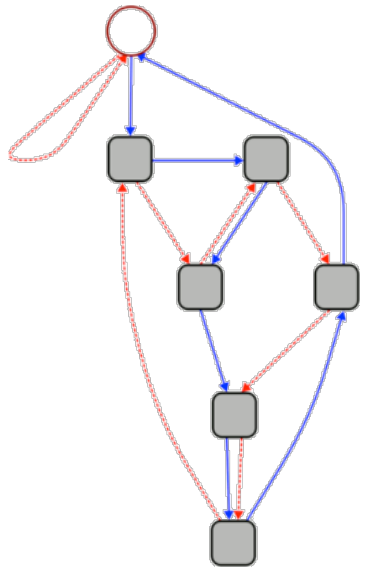


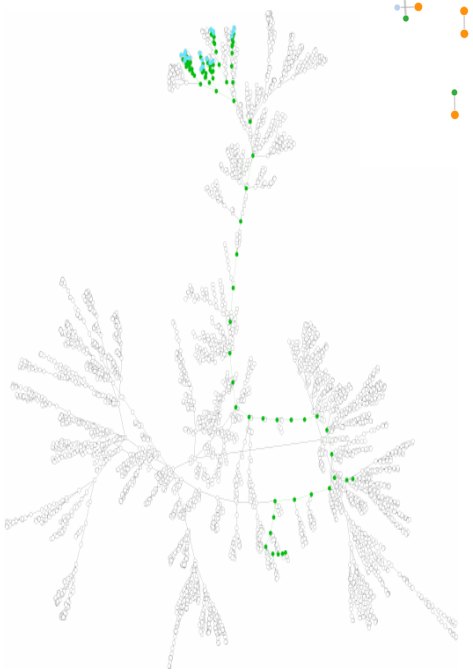
HAMLET

TROILUS AND CRESSIDA





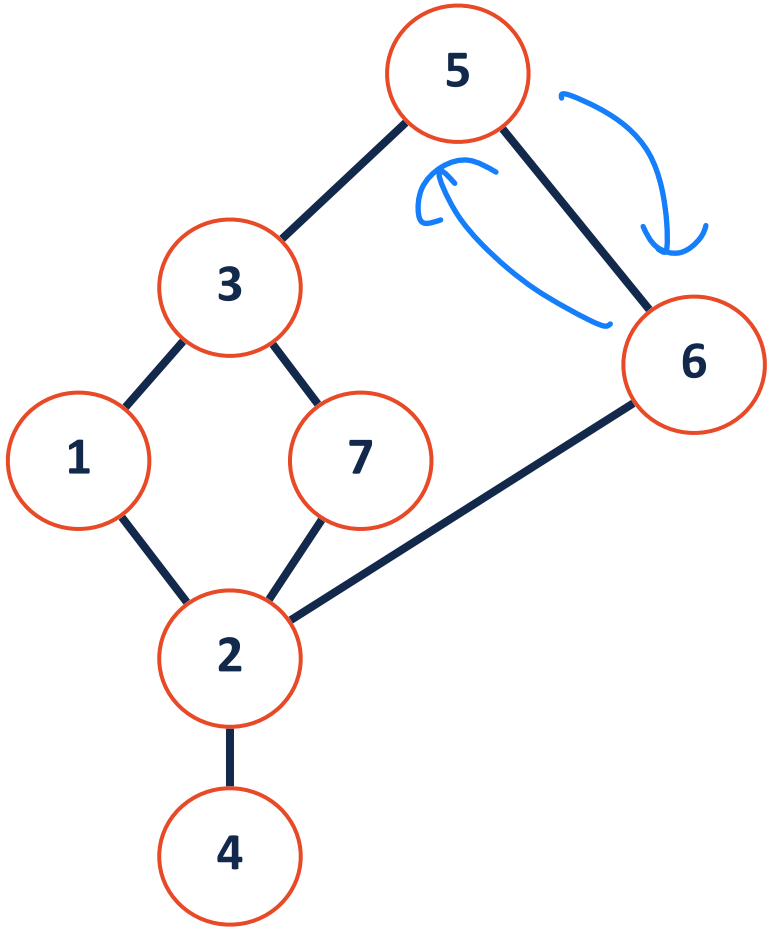




Graph Vocabulary

$G = (V, E)$

A **graph** is a data structure containing a set of vertices and a set of edges



Vertex: Nodes of the graph

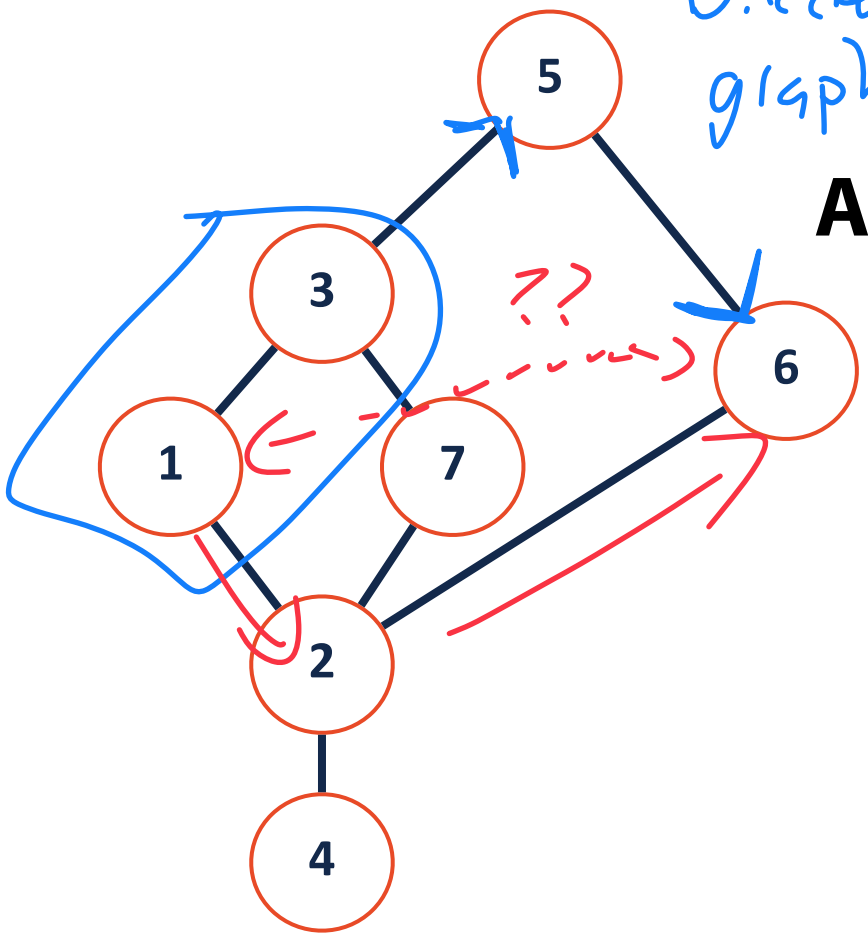
↳ can have key, value, ...

Edges: The connections between nodes

Defined by two endpoints

(5, 6) ||
(6, 5)

Graph Vocabulary



Degree: # of edges touching a vertex

Directed graph
- In-Degree — # incoming incident edges
- Out-Degree — # outgoing edges

Adjacency: Two vertices are adjacent if they are connected by an edge

- (1,3) is adjacent
- (1,6) not adjacent

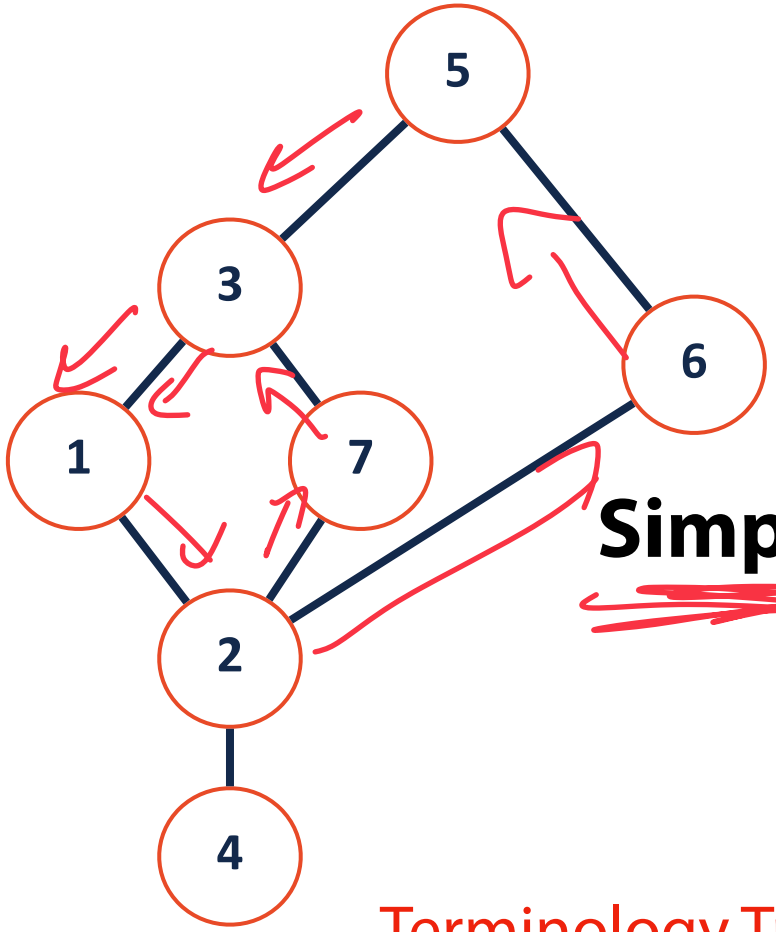
Path: A sequence of vertices (or edges) between two nodes

By vertex: 1 → 2 → 6

edge: (1,2), (2,6)

Graph Vocabulary

A graph has **no root** and **may contain cycles**



Cycle: A path from a node to itself

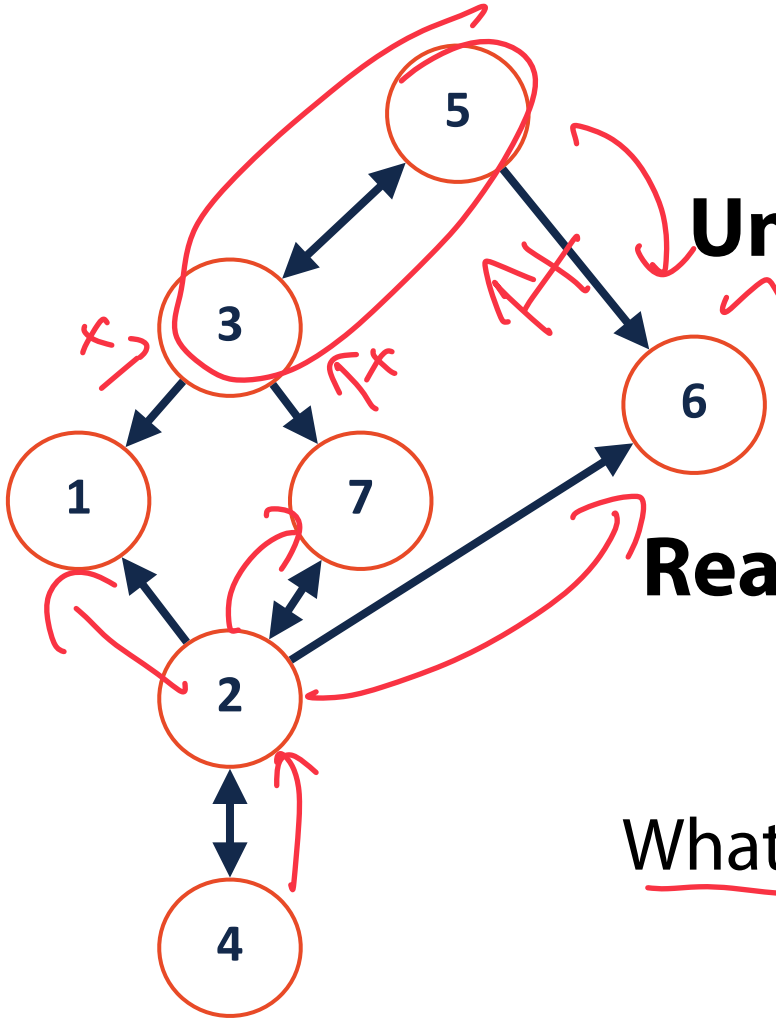
Simple Graph: No self-loops or multi-edges

Terminology Trivia: Every tree is a graph but not every graph is a tree

Graph Vocabulary

Join Code: 225

A graph may be **directed** or **undirected**



Directed: Edges are one way connections

Undirected: Traversable in either direction

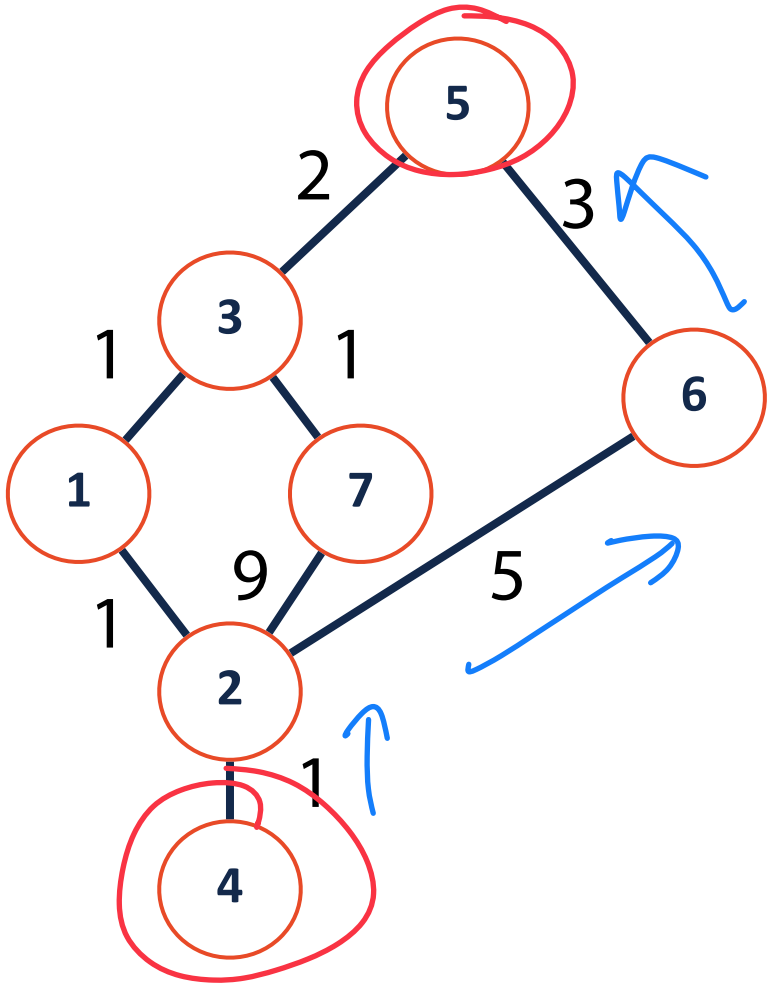
Reachability: v_2 is reachable from v_1 if there is a path from v_1 to v_2

What nodes are not reachable from 4?

Graph Vocabulary

Join Code: 225

A graph may be **weighted** or **unweighted**



Weights: A value associated with an edge

What is the shortest path from 4 to 5?

4 → 2 → 1 → 3 - 5

Smallest weight

If unweighted: 4 - 2 - 6 - 5

↳ minimum edge count

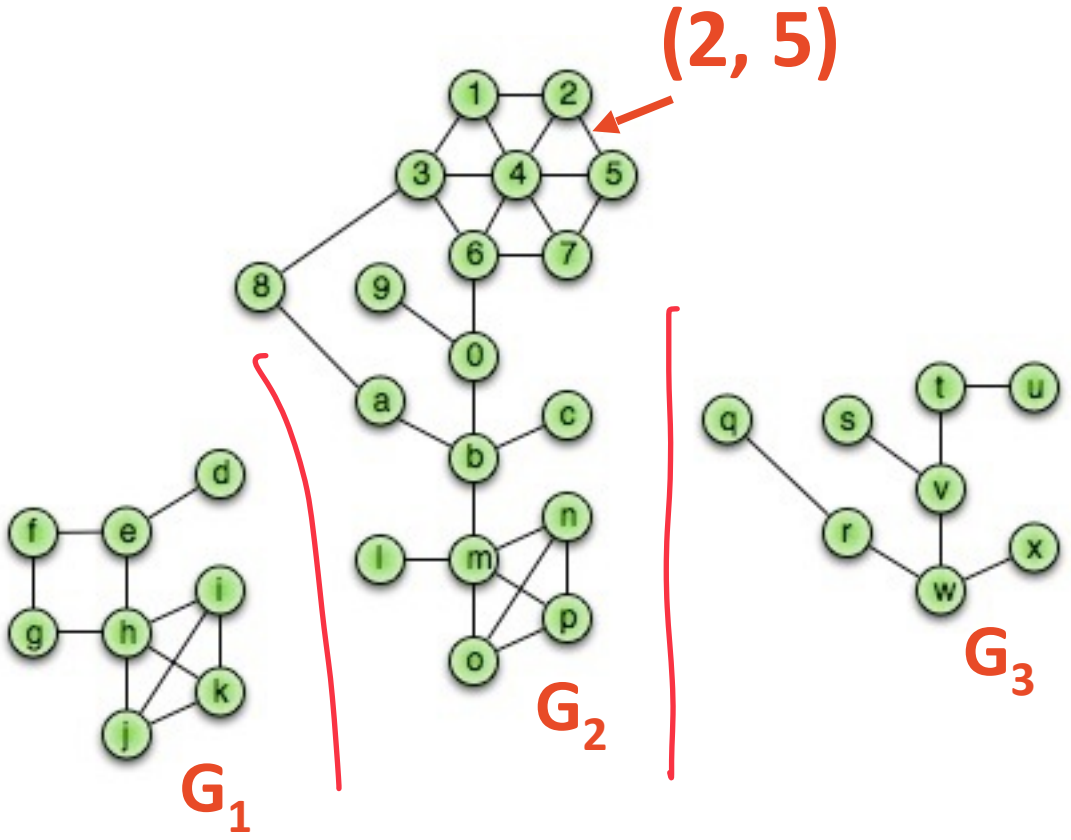
(3)

Graph Vocabulary

$G = (V, E)$ ← set of edges

$|V| = n$ ← set of vertices

$|E| = m$



Subgraph(G):

G' = (V', E'):

V' ∈ V, E' ∈ E, and

(u, v) ∈ E' → u ∈ V', v ∈ V'

Graph Vocabulary

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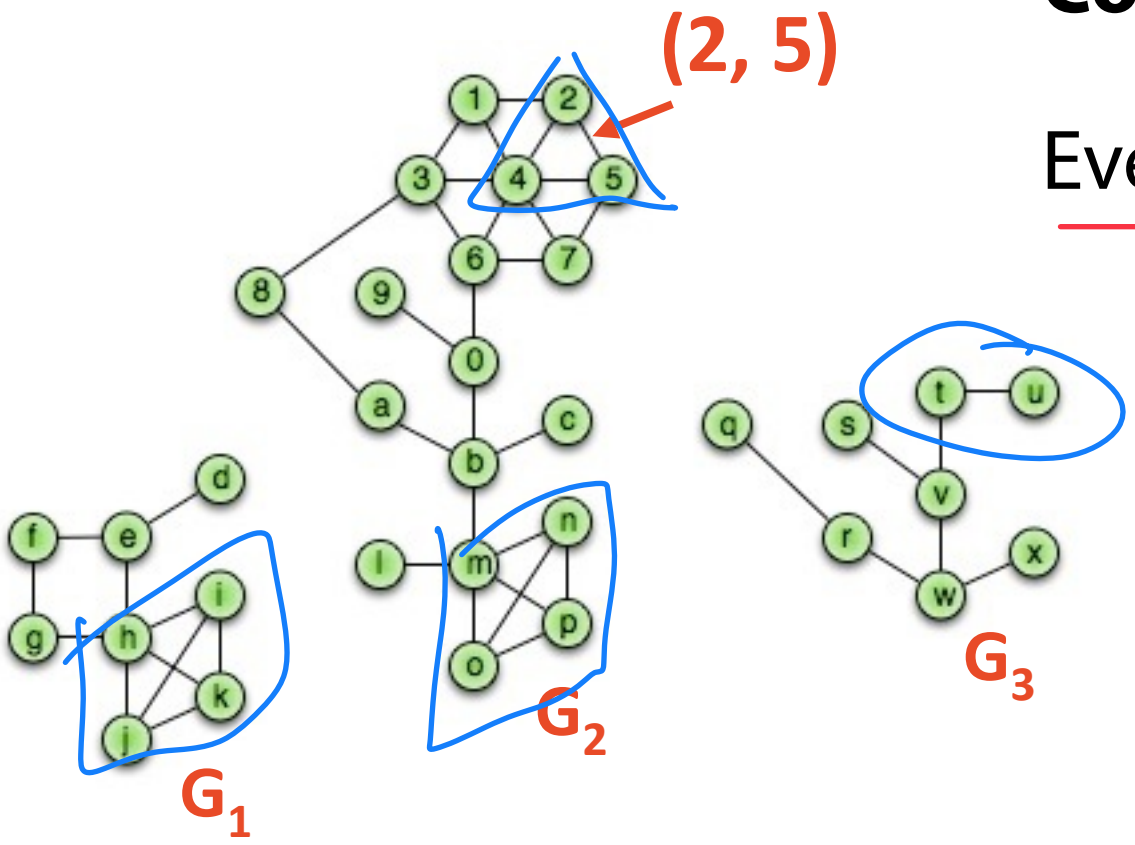

Complete Subgraph:

direct edge

Every pair of vertices are adjacent

A B

Graph Vocabulary

$G = (V, E)$

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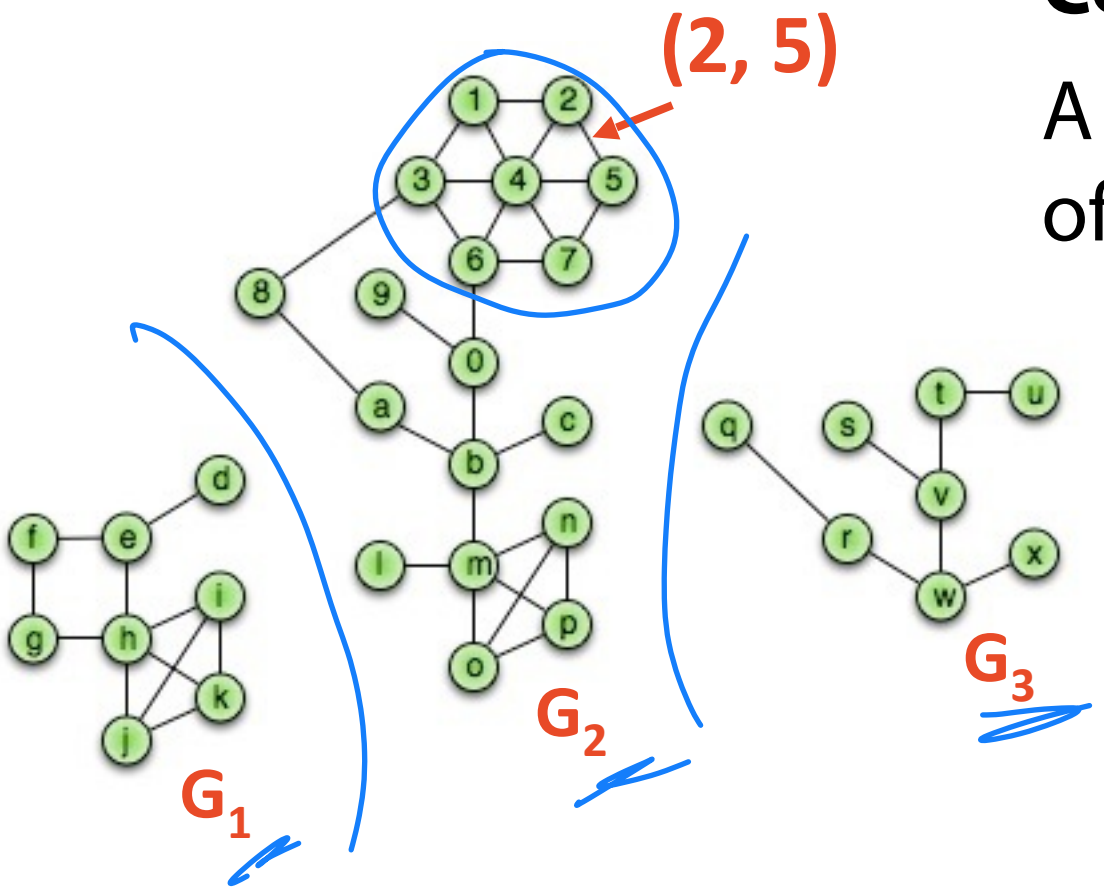

Subgraph(G):

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Connected Subgraph:

A path exists between every pair of vertices

Graph Vocabulary

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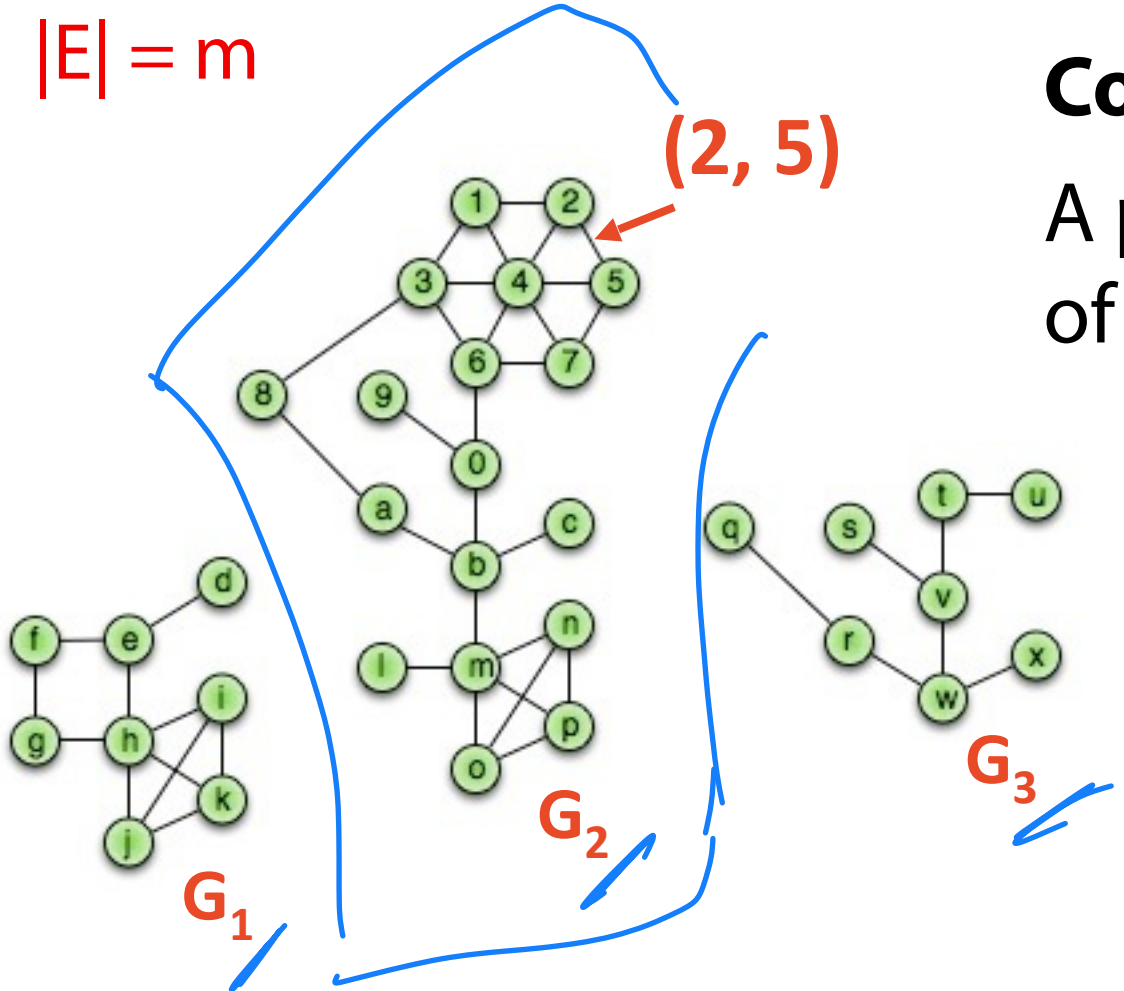


Subgraph(G):

$G' = (V', E')$:

$V' \in V$, $E' \in E$, and

$(u, v) \in E' \rightarrow u \in V', v \in V'$

Connected Subgraph:

A path exists between every pair of vertices $V = \{9, 0, 6, 7\}$

Connected Components:

A connected subgraph that is not part of a larger subgraph

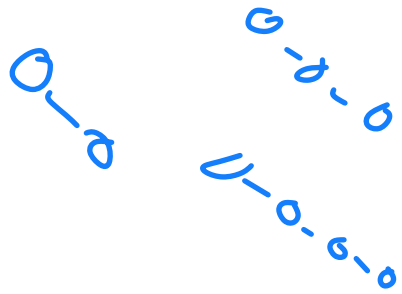
$V = \{G_2\}$

Graph Vocabulary

$G = (V, E)$

$|V| = n$

$|E| = m$



Subgraph(G):

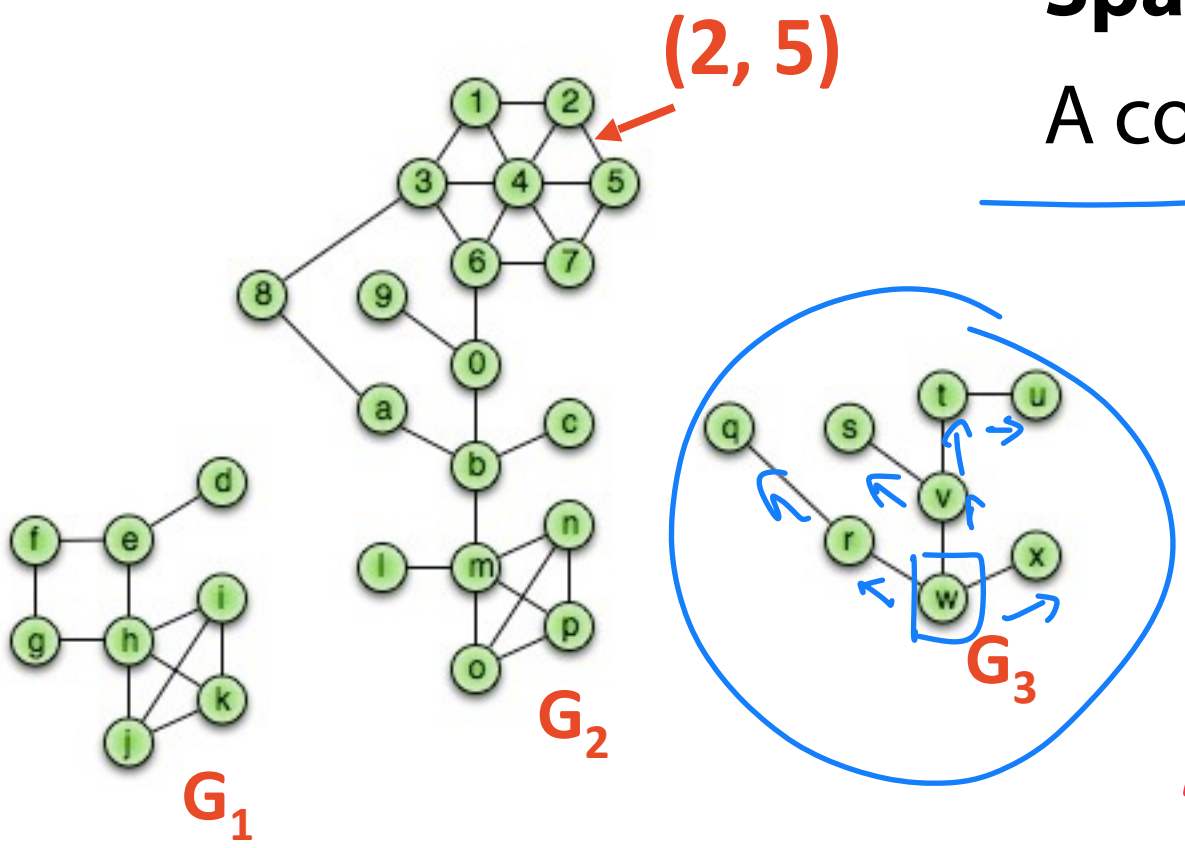
G' = (V', E'):

V' ∈ V, E' ∈ E, and

(u, v) ∈ E' → u ∈ V', v ∈ V'

Spanning Tree:

A connected graph with no cycles

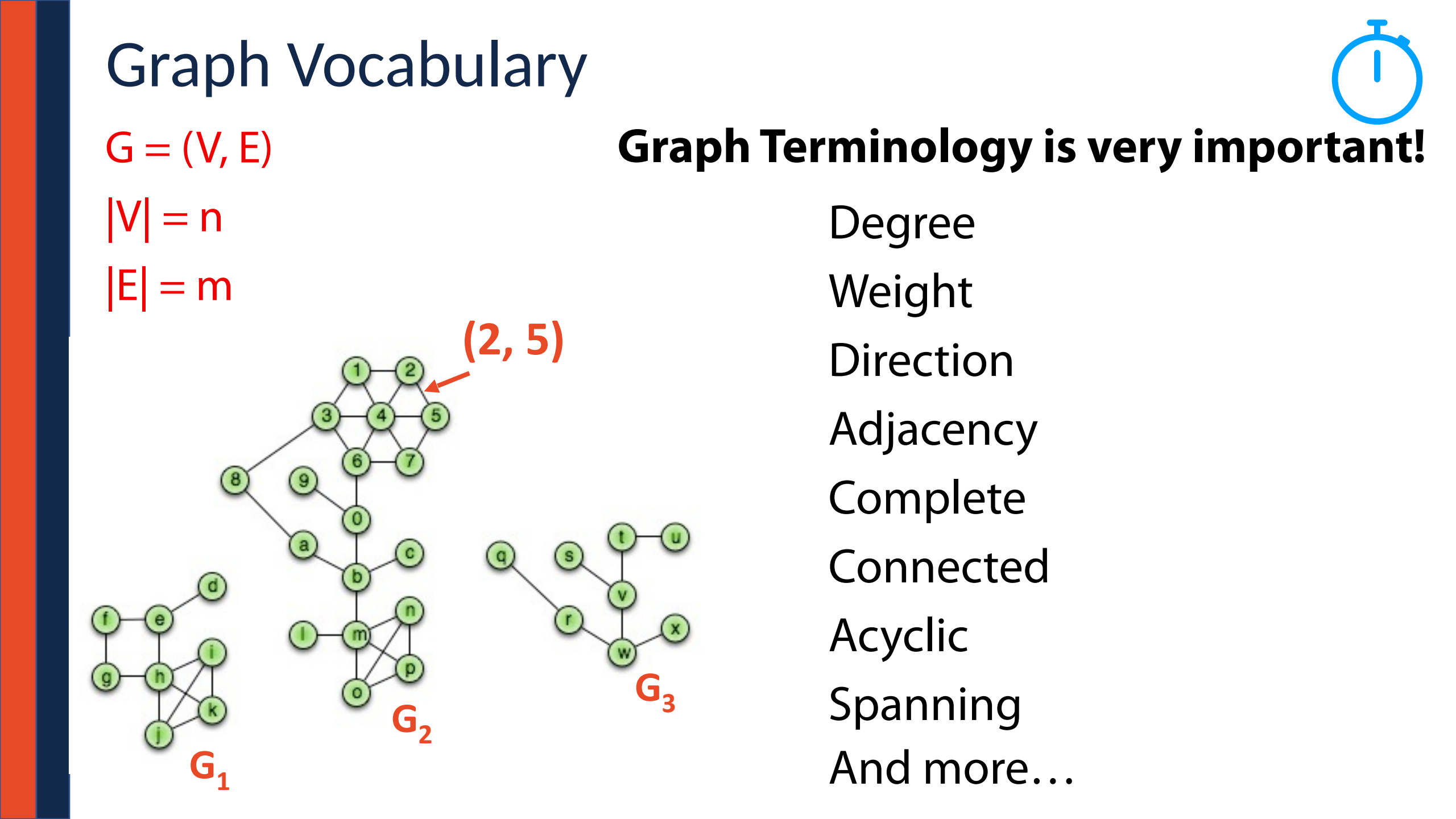


↳ # of edges to produce a spanning tree if graph has |V|=n?

n−1 edges to connect n vertices

Minimum spanning tree is min weight cost to build tree

Graph Vocabulary



$G = (V, E)$

$|V| = n$

$|E| = m$

Graph Terminology is very important!

- Degree
- Weight
- Direction
- Adjacency
- Complete
- Connected
- Acyclic
- Spanning

And more…

Running times are often reported by **n**, the number of vertices, but often depend on **m**, the number of edges.

Whats the relationship between **n** and **m**?

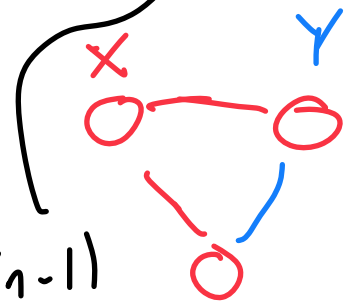
Join Code: 225

$n-1 \leq m \leq O(n^2)$

Minimum Edges:

Unconnected Graph: 0 (zero)

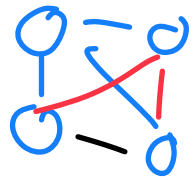
Connected (Simple) Graph: $n-1$ edges at minimum



Maximum Edges:

Connected (Simple) Graph: $(n-1) + (n-2) + \dots = \frac{n(n-1)}{2}$

$$\sum_{v \in V} deg(v) = O(n^2)$$



Graphs

Given a collection of individual DMs between individuals, you want to build a graph of connections in a social network.

What is a vertex?

↳ an individual

What is an edge?

↳ a DM (or follow)

Are the edges directed or undirected?

↳ Directed (but you choose!)

Are the edges weighted or unweighted?

↳ You choose!

Use case matters!

Graphs

Given a collection of roads between cities in Illinois, you want to build a graph of the transportation infrastructure in the state.

What is a vertex?

What is an edge?

Are the edges directed or undirected?

Are the edges weighted or unweighted?

Graphs

It is important to be able to describe the structure of a graph given input.

Some other common questions:

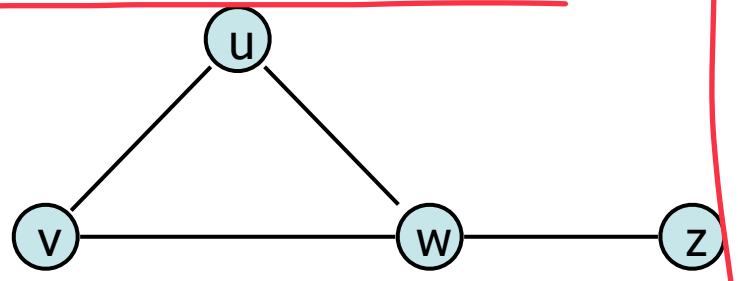
- Does your graph have cycles?
- What is the largest / smallest / average degree in your graph?
- What is the total number of edges?
- …

Of course, we also have to understand the graph as a **data structure**

Graph Implementation

What information do we need to store to fully define a graph?

Vertex:



Edge:

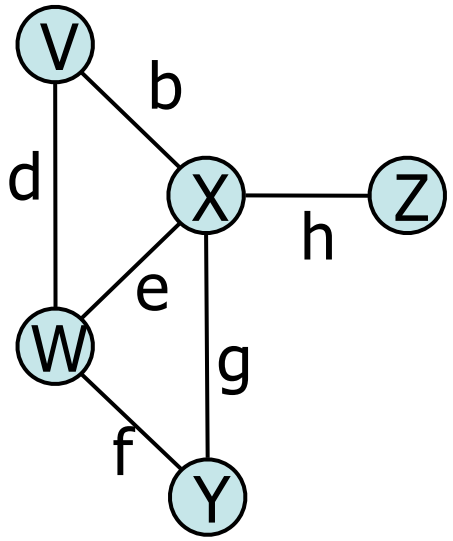
What information do we want to be able to find out quickly?

What operations do we want to prioritize?

Graph ADT

Data:

- Vertices
- Edges
- Some data structure maintaining the structure between vertices and edges.



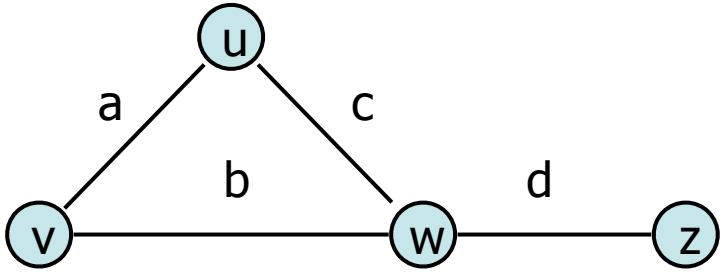
Functions:

- insertVertex(K key);
- insertEdge(Vertex v1, Vertex v2, K key);

- removeVertex(Vertex v);
- removeEdge(Vertex v1, Vertex v2);

- getEdges(Vertex v);
- areAdjacent(Vertex v1, Vertex v2);
- origin(Edge e);
- destination(Edge e);

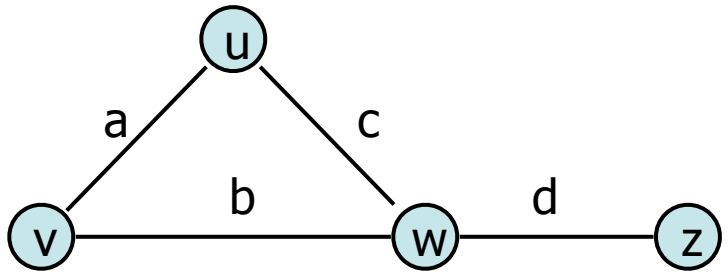
Graph Implementation Idea

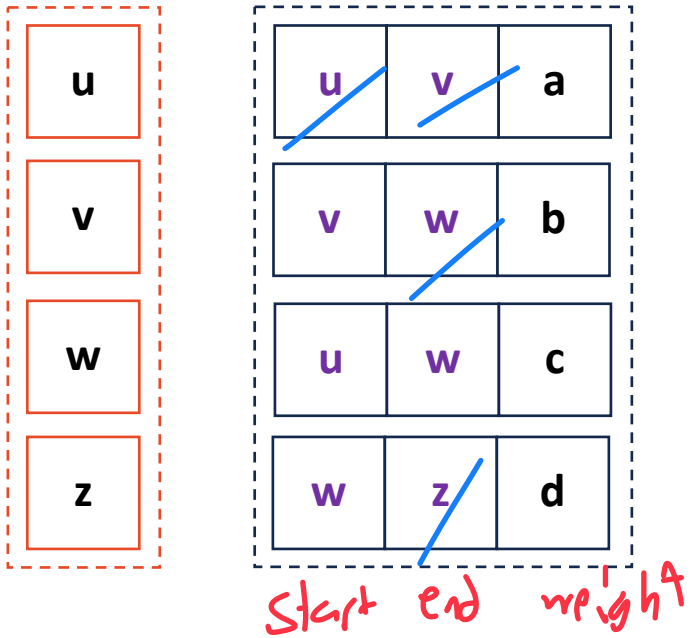


↳ why not list?

Graph Implementation: Edge List $|V| = n, |E| = m$

The equivalent of an 'unordered' data structure





Vertex Storage:

(optional) ↳ a list or array of vertices

Can get vertex list from edge list

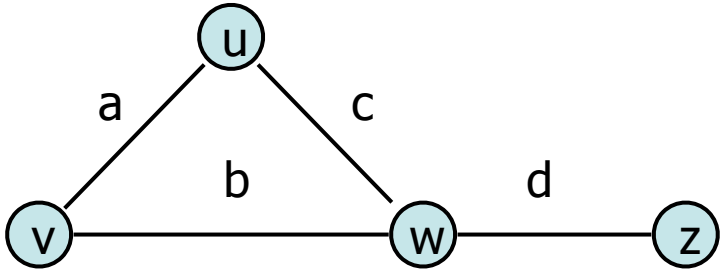
Edge Storage:

↳ A unsorted array of start, stop, weight

(!) great storage potential!

Graph Implementation: Edge List

$|V| = n, |E| = m$



getEdges(Vertex v) $O(m)$

↳ walk through entire edge list

O(n)

u
v
w
z

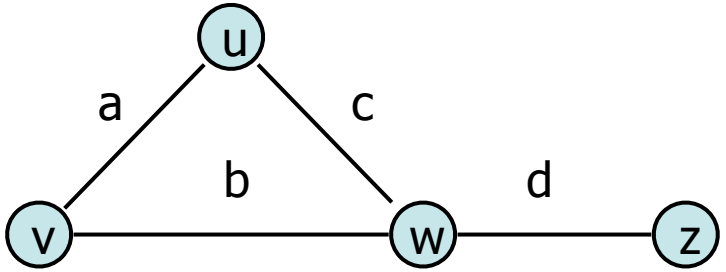
O(m)

u	v	a
v	w	b
u	w	c
w	z	d

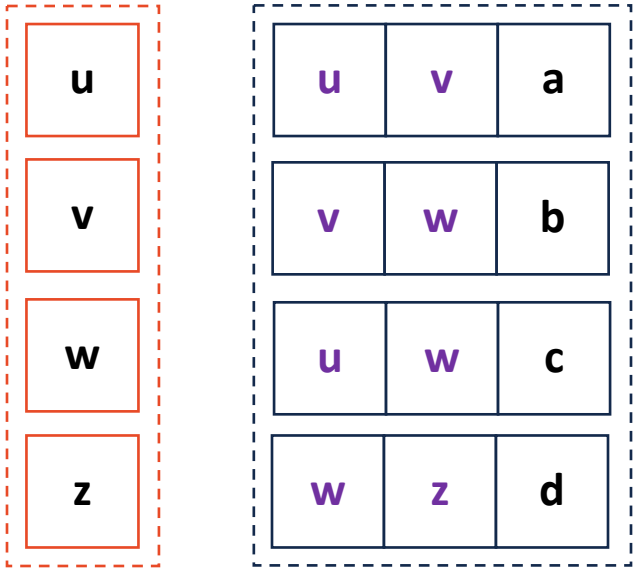
areAdjacent(Vertex v1, Vertex v2) $O(m)$

v1 v2

Graph Implementation: Edge List |V|= n, |E|= m

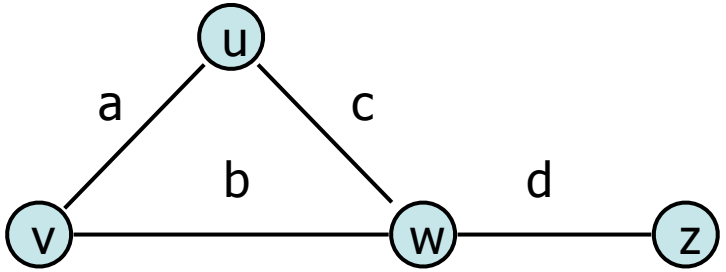


insertVertex(K key)

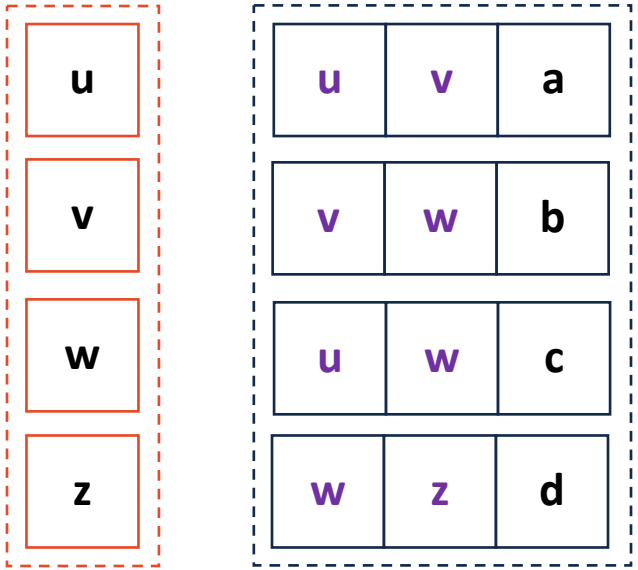


removeVertex(Vertex v)

Graph Implementation: Edge List |V|= n, |E|= m



insertEdge(Vertex v1, Vertex v2, K key)



removeEdge(Vertex v1, Vertex v2)

Graph Implementation: Edge List

Pros:

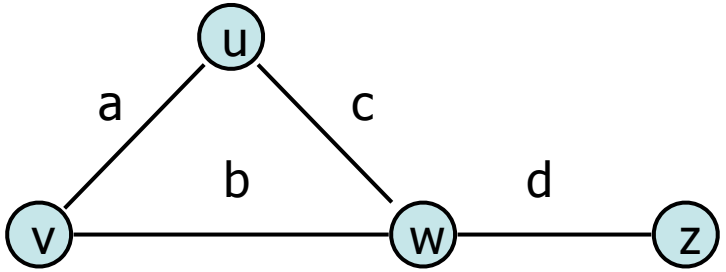
Cons:

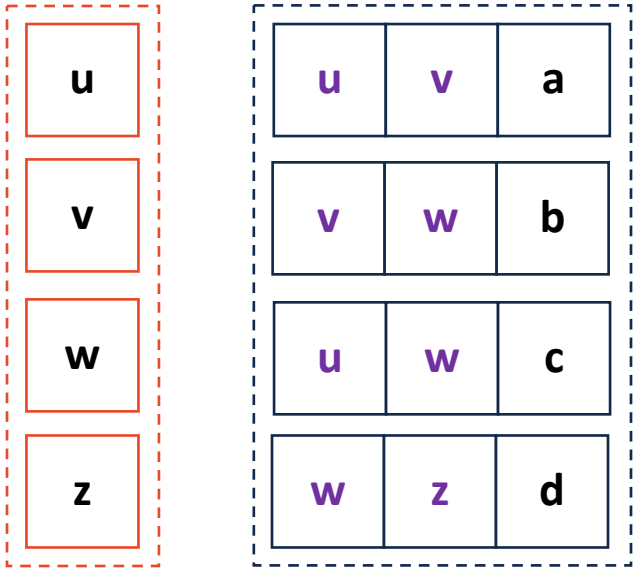
Graph Implementation: Brainstorming better

What operations might I want to do very quickly?

What modifications might allow me to do these things faster?

Graph Implementation: Adjacency Matrix





	u	v	w	z
u				
v				
w				
z				