String Algorithms and Data Structures

Approximate Pattern Matching

CS 199-225
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April 13, 2026

UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Department of Computer Science

Last

"main" lecture!

^_^

Learning Objectives

Review approximate pattern matching

Formalize edit distance storage as an 'edit string'

Discuss strategies for efficient APM with edits

Introduce dynamic programming

Approximate Pattern Matching

Input: A text *T*, a pattern *P*, and a distance *d*

Output: All positions in *T* where *P* has at most *d* mismatches or edits

P: `word`

T: `There would have been a time for such a word`

Alignment 1: `word`

Alignment 2: `word`

~~Not a match!~~

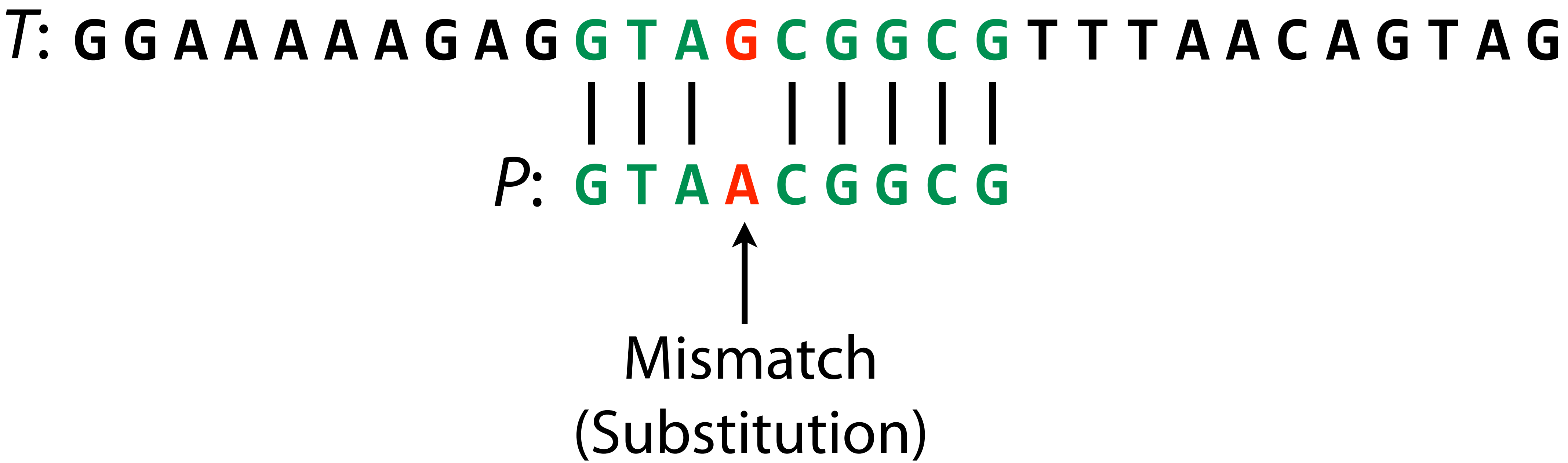
Match!

Distance 2 match!

Distance 0 match!

Hamming Distance

The number of **substitutions** required to turn one string into another



```
T: GGAAAAAGAGGTAGCGGCGTTTAACAGTAG
             ||| |||||
P:           GTAACGGCG
```

Mismatch
(Substitution)

Approximate Pattern Matching

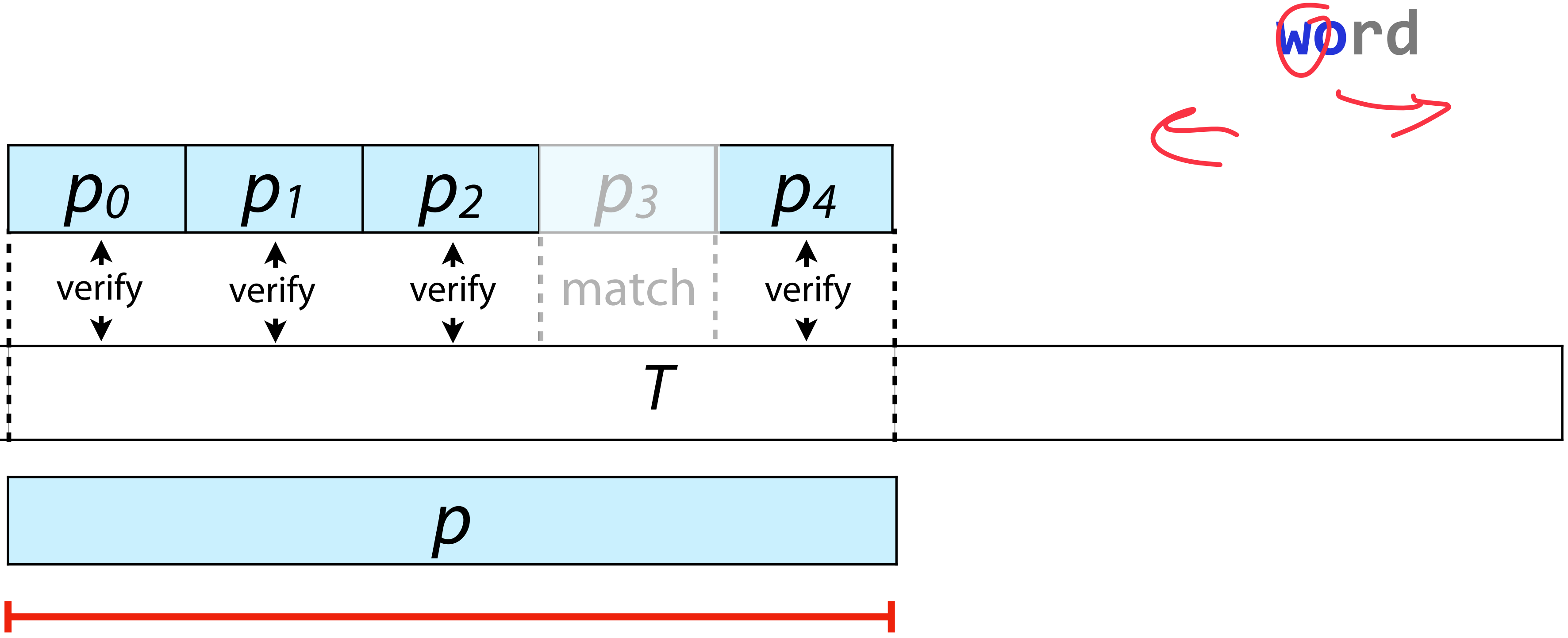
As a ***heuristic***, seed and extend reduces the overall search space

T: `There would have been a time for such a word`

`word`

`word`

`word`



Only consider mismatches while verifying a seed hit

Edit Distance

```
Score =  248 bits (129), Expect = 1e-63
Identities = 213/263 (80%), Gaps = 34/263 (12%)
Strand = Plus / Plus
```

Substitution

```
Query: 161 atatcaccacgtcaaaggtgactccaactcca---ccactccattttgttcagataatgc 217
           |||||||||||||||||||||||||||||  |     | |   || ||||||||||||||
Sbjct: 481 atatcaccacgtcaaaggtgactccaact-tattgatagtgttttatgttcagataatgc 539

Query: 218 ccgatgatcatgtcatgcagctccaccgattgtgagaacgacagcgacttccgtcccagc 277
           |||||||   ||||||||||||||||||||| || |            ||||||||||||
Sbjct: 540 ccgatgactttgtcatgcagctccaccgattttg-g------------ttccgtcccagc 586
```

Deletion

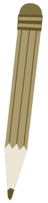
```
Query: 278 c-gtgcc--aggtgctgcctcagattcaggttatgccgctcaattcgctgcgtatatcgc 334
           |  || |  | ||||||||||||||||||||||||||||||||||||||| |||||||||
Sbjct: 587 caatgacgta-gtgctgcctcagattcaggttatgccgctcaattcgctgggtatatcgc 645

Query: 335 ttgctgattacgtgcagctttcccttcaggcggga------------ccagccatccgtc 382
           |||||||||||||||||||||||||||||||||||            |||||||||||||
Sbjct: 646 ttgctgattacgtgcagctttcccttcaggcgggattcatacagcggccagccatccgtc 705
```

Insertion

```
Query: 383 ctccatatc-accacgtcaaagg 404
            |||||||| |||||||||||||
Sbjct: 706 atccatatcaaccacgtcaaagg 728
```

Edit Distance

Imagine edits are introduced by an *optimal editor* working left-to-right:



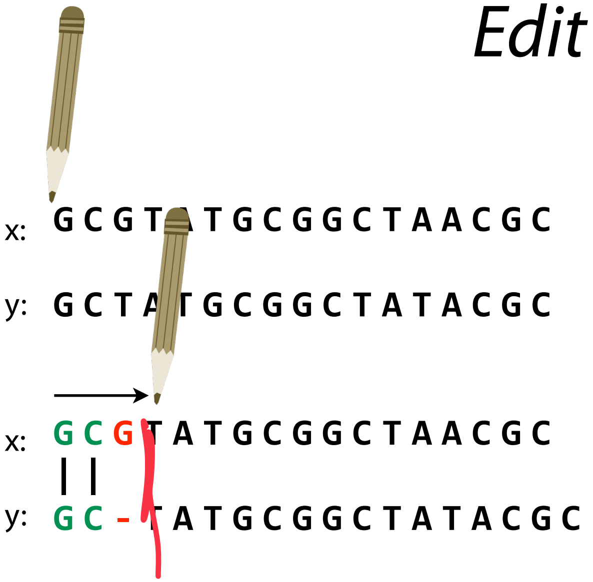
x: **GCGTATGCGGCTAACGC**

y: **GCTATGCGGCTATACGC**

Edit Distance

Imagine edits are introduced by an *optimal editor* working left-to-right:

Edit string summarizes how editor turns *x* into *y:*



```
x:  GCGTATGCGGCTAACGC
y:  GCTATGCGGCTATACGC

x:  GCGTATGCGGCTAACGC
    ||
y:  GC-TATGCGGCTATACGC
```

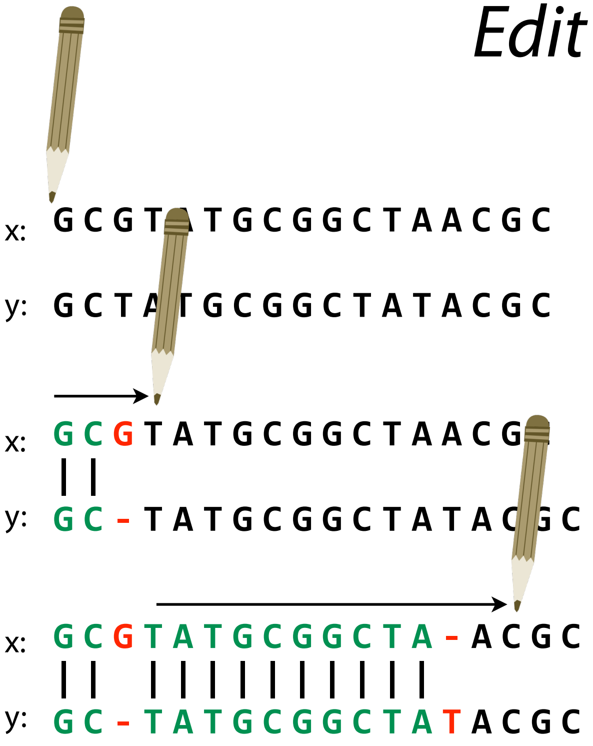
M M R

Operations:
M = match, **R** = replace (substitute),
I = insert into *x*, **D** = delete from *x*

Edit Distance

Imagine edits are introduced by an *optimal editor* working left-to-right:

Edit string summarizes how editor turns *x* into *y:*



```
x:  G C G T A T G C G G C T A A C G C

y:  G C T A T G C G G C T A T A C G C

x:  G C G T A T G C G G C T A A C G C
    | |
y:  G C - T A T G C G G C T A T A C G C

x:  G C G T A T G C G G C T A - A C G C
    | |   | | | | | | | | | |
y:  G C - T A T G C G G C T A T A C G C
```

Operations:

M = match, **R** = replace (substitute),

I = insert into *x*, **D** = delete from *x*

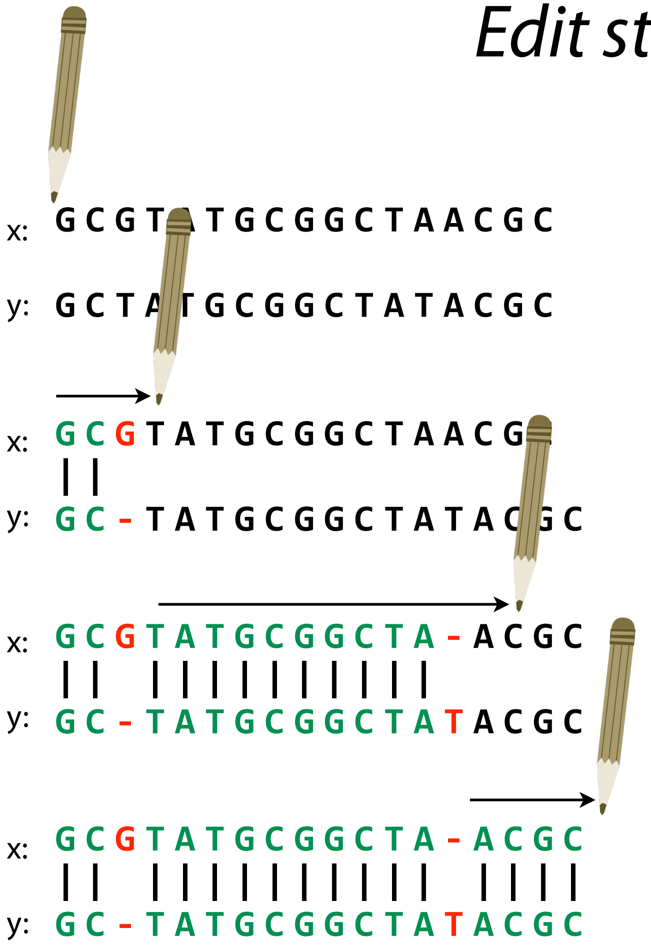
`MMD`

Reminder: this is **D**, not **I**, because we have to delete a character from *x* to make it more like *y*

Edit Distance

Imagine edits are introduced by an *optimal editor* working left-to-right:

Edit string summarizes how editor turns *x* into *y:*



```
x: GCGTATGCGGCTAACGC

y: GCTATGCGGCTATACGC
```

```
x: GCGTATGCGGCTAACGC
   ||
y: GC-TATGCGGCTATACGC
```

```
x: GCGTATGCGGCTA-ACGC
   || ||||||||||
y: GC-TATGCGGCTATACGC
```

```
x: GCGTATGCGGCTA-ACGC
   || |||||||||| ||||
y: GC-TATGCGGCTATACGC
```

Operations:

M = match, **R** = replace (substitute),

I = insert into *x*, **D** = delete from *x*

MMD

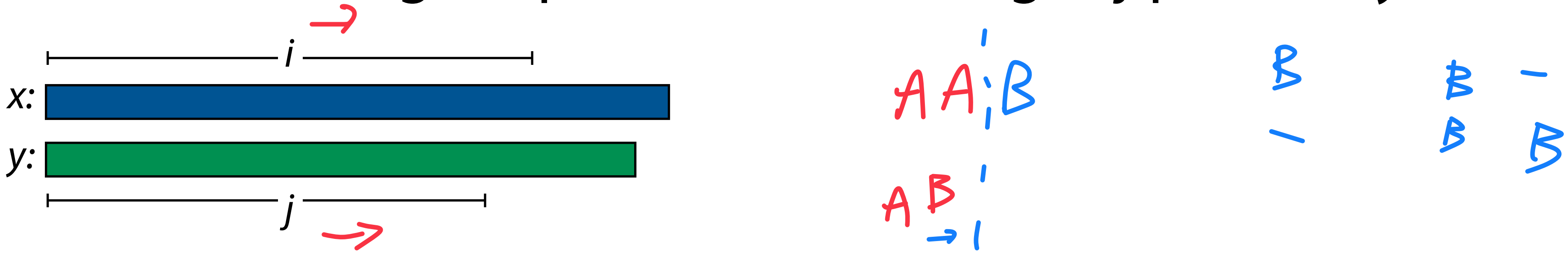
Reminder: this is **D**, not **I**, because we have to delete a character from *x* to make it more like *y*

MMDMMMMMMMMMMI

MMDMMMMMMMMMMIMMMM

Edit Distance

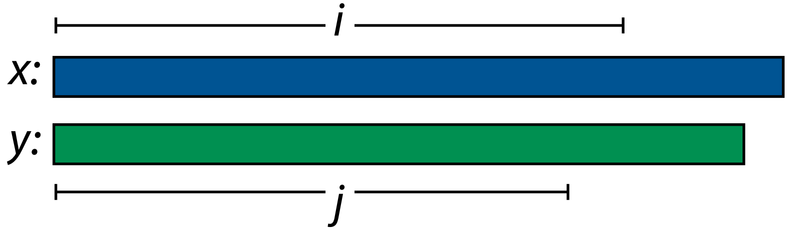
D[*i*, *j*]: edit distance between length-*i* prefix of *x* and length-*j* prefix of *y*



Optimal edit string for D[*i*, *j*] is built by ***extending a shorter optimal string by 1 operation***. 3 options:

Edit Distance

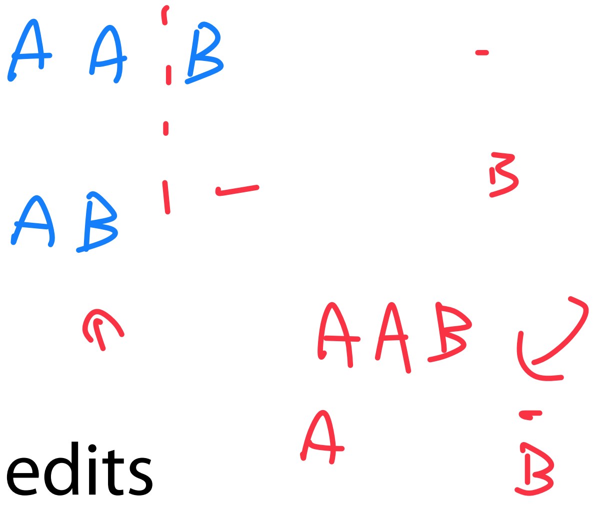
D[*i*, *j*]: edit distance between length-*i* prefix of *x* and length-*j* prefix of *y*



Optimal edit string for D[*i*, *j*] is built by ***extending a shorter optimal string by 1 operation***. 3 options:

- Append **D** to transcript for D[*i*-1, *j*]
- Append **I** to transcript for D[*i*, *j*-1]
- Append **M** or **R** to transcript for D[*i*-1, *j*-1]

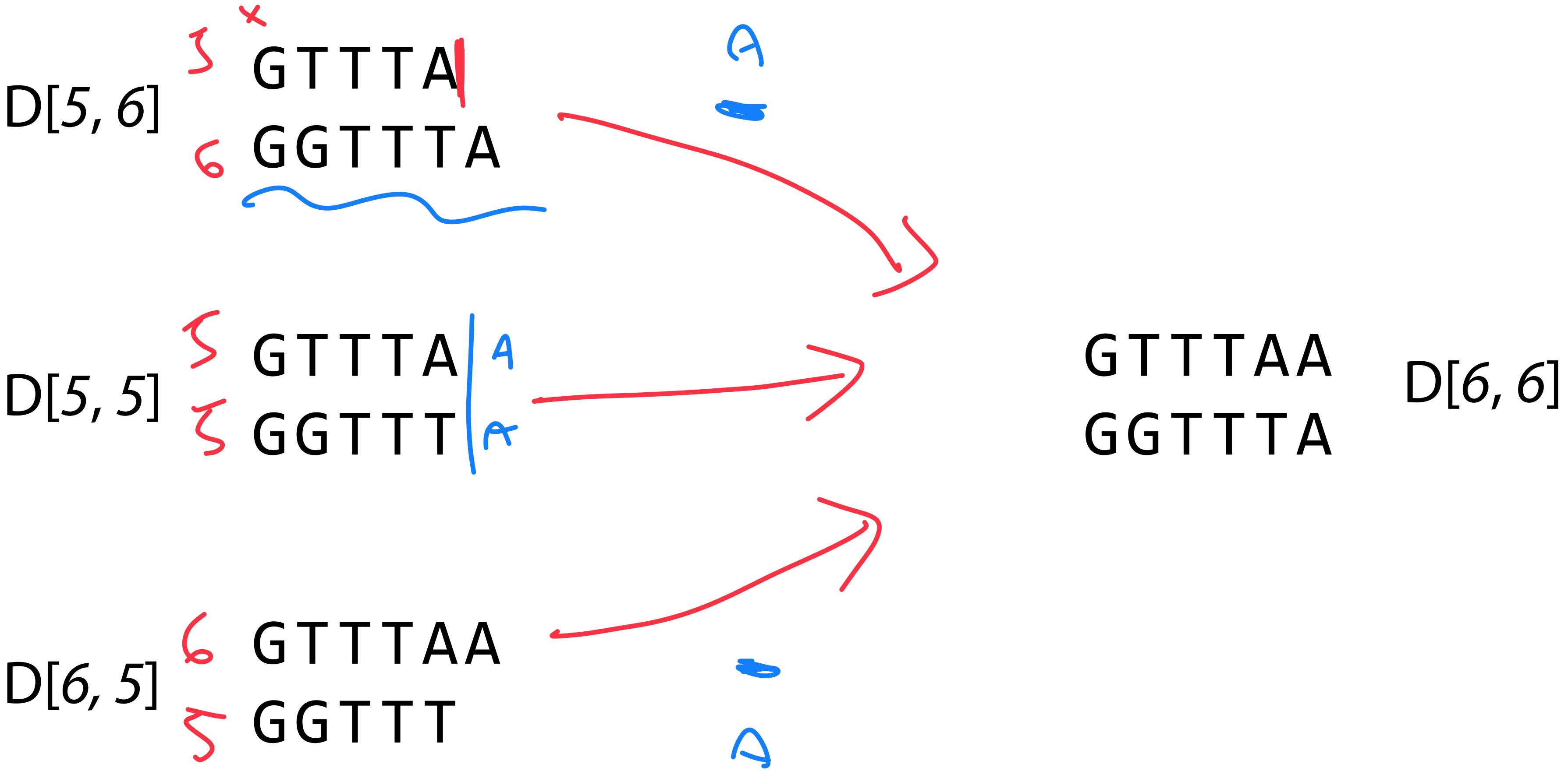
We choose based on whichever option has the fewest edits



Edit Distance

X: GTTTAA

Y: GGTTTA



D[5, 6]

x

5 GTTTA

6 GGTTTA

A

—

D[5, 5]

5 GTTTA A

5 GGTTT A

D[6, 5]

6 GTTTAA

5 GGTTT

—

A

GTTTAA

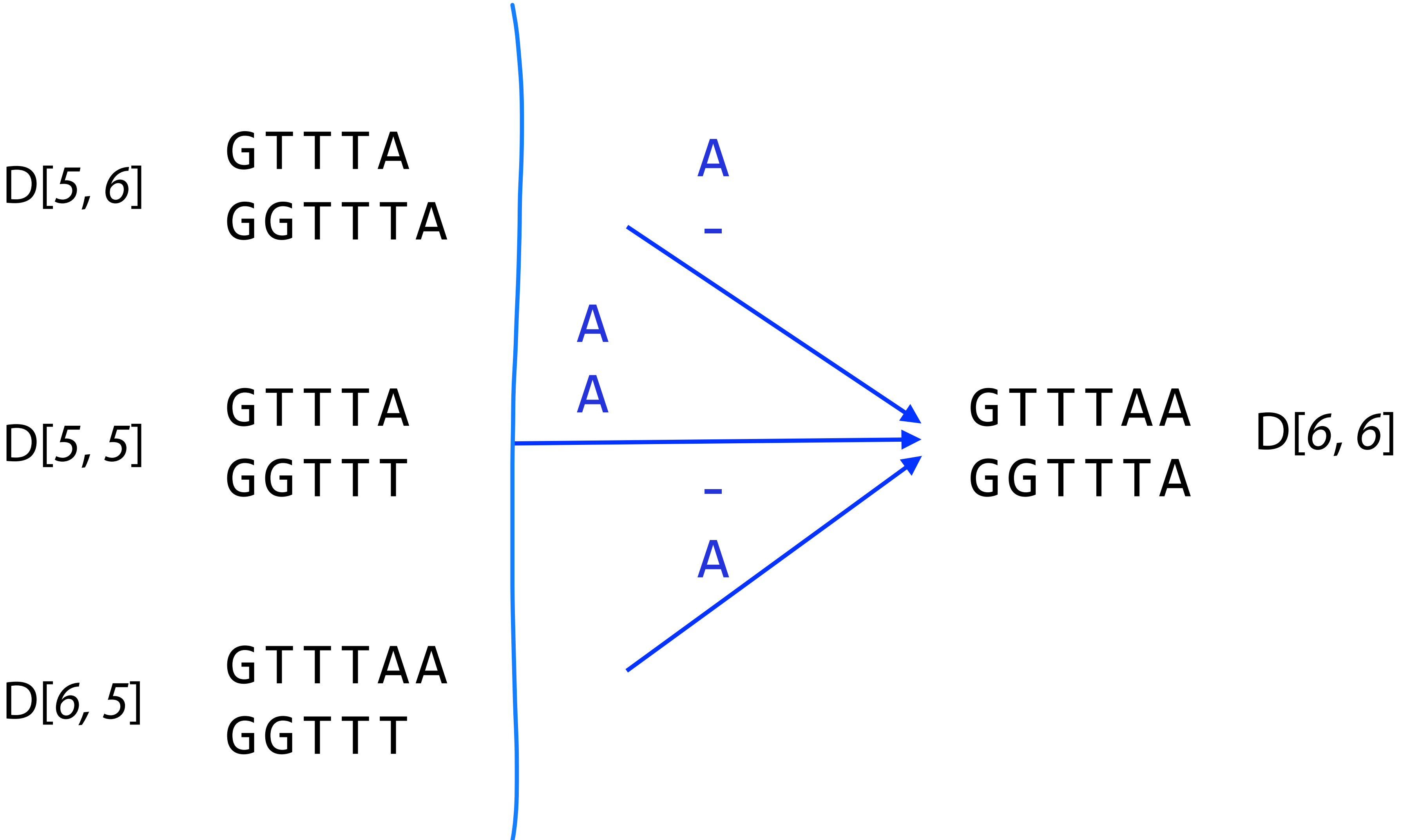
GGTTTA

D[6, 6]

Edit Distance

X: GTTTAA

Y: GGTTTA



D[*5*, *6*]
GTTTA
GGTTTA

D[*5*, *5*]
GTTTA
GGTTT

D[*6*, *5*]
GTTTAA
GGTTT

GTTTAA
GGTTTA
D[*6*, *6*]

Edit Distance

X: GTTTAA Y: GGTTTA

D[*5*, *6*]

```
GTTTA
GGTTTA
```

D[*5*, *5*]

```
GTTTA
GGTTT
```

D[*6*, *5*]

```
GTTTAA
GGTTT
```

Edit Distance

X: GTTTAA

Y: GGTTTA

MIMMMM

D[*5*, *6*]

G	-	T	T	T	A
G	G	T	T	T	A

D[*5*, *5*]

GTTTA
GGTTT

D[*6*, *5*]

GTTTAA
GGTTT

Edit Distance

X: GTTTAA

Y: GGTTTA

MIMMMM

D[*5*, *6*]

G	-	T	T	T	A
G	G	T	T	T	A

MRMMR

D[*5*, *5*]

G	T	T	T	A
G	G	T	T	T

D[*6*, *5*]

GTTTAA
GGTTT

Edit Distance

X: GTTTAA

Y: GGTTTA

MIMMMM

D[*5*, *6*]

G	-	T	T	T	A
G	G	T	T	T	A

MRMMR

D[*5*, *5*]

G	T	T	T	A
G	G	T	T	T

MRMMRD

D[*6*, *5*]

G	T	T	T	A	A
G	G	T	T	T	-

Edit Distance

X: GTTTAA

Y: GGTTTA

MIMMMM

D[5, 6]

G	-	T	T	T	A
G	G	T	T	T	A

G	-	T	T	T	A	A
G	G	T	T	T	A	-

D[6, 6]^{1}

MRMMR

D[5, 5]

G	T	T	T	A
G	G	T	T	T

G	T	T	T	A	A
G	G	T	T	T	A

D[6, 6]^{2}

MRMMRD

D[6, 5]

G	T	T	T	A	A
G	G	T	T	T	-

G	T	T	T	A	A	-
G	G	T	T	T	-	A

Dont do this

D[6, 6]^{3}

Edit Distance

X: GTTTAA

Y: GGTTTA

$D[5, 6]$

MIMMMM

G	-	T	T	T	A
G	G	T	T	T	A

MIMMMMD

G	-	T	T	T	A	A
G	G	T	T	T	A	-

$D[6, 6]^1$

$D[5, 5]$

MRMMR

G	T	T	T	A
G	G	T	T	T

MRMMRM

G	T	T	T	A	A
G	G	T	T	T	A

$D[6, 6]^2$

$D[6, 5]$

MRMMRD

G	T	T	T	A	A
G	G	T	T	T	-

MRMMRDI

G	T	T	T	A	A	-
G	G	T	T	T	-	A

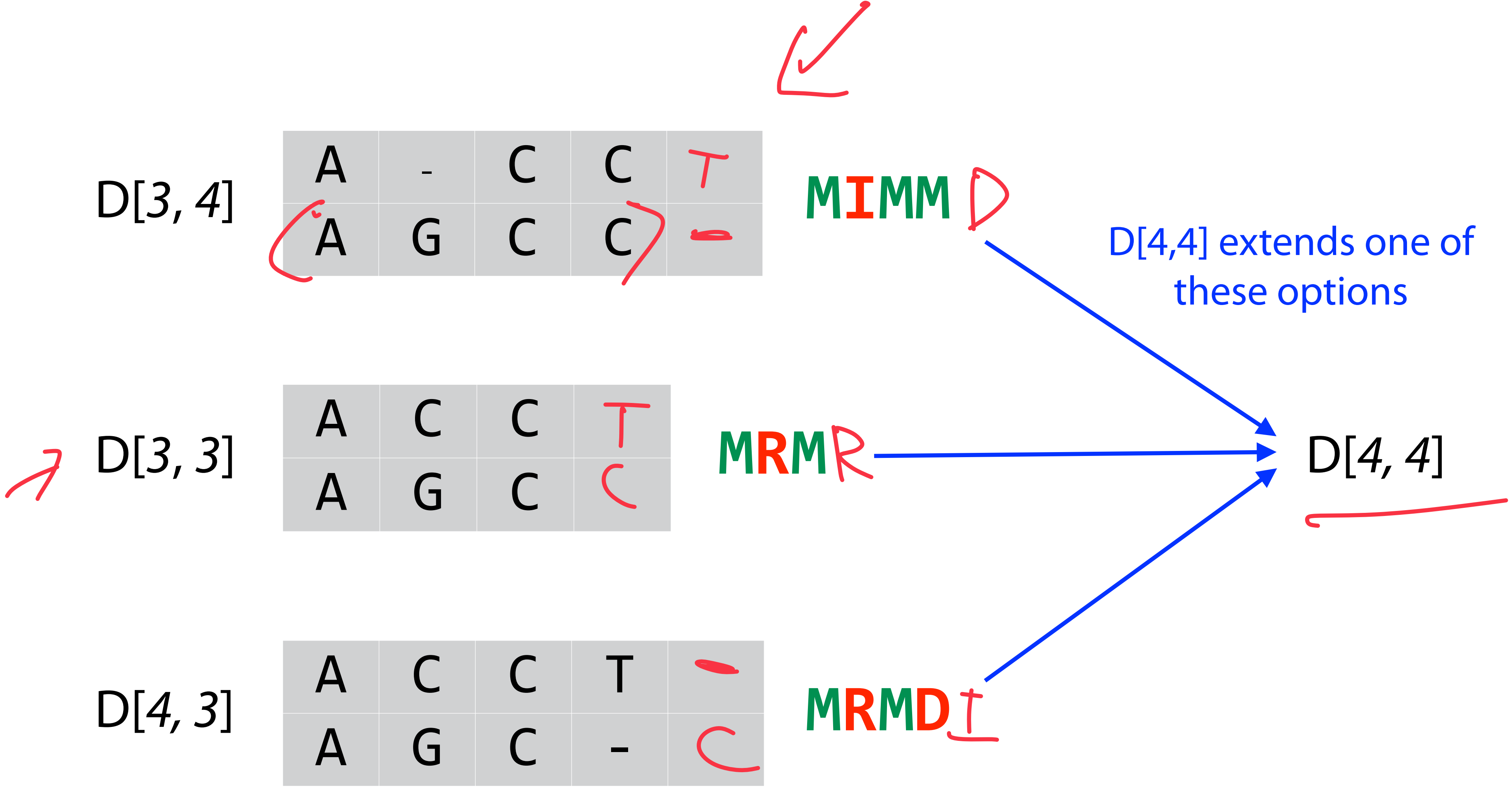
$D[6, 6]^3$

Edit Distance

Join Code: 225

X: ACCT

Y: AGCC



D[3, 4]

A	-	C	C	T
A	G	C	C	-

MIMMD

D[3, 3]

A	C	C	T
A	G	C	C

MRMR

D[4, 3]

A	C	C	T	-
A	G	C	-	C

MRMDI

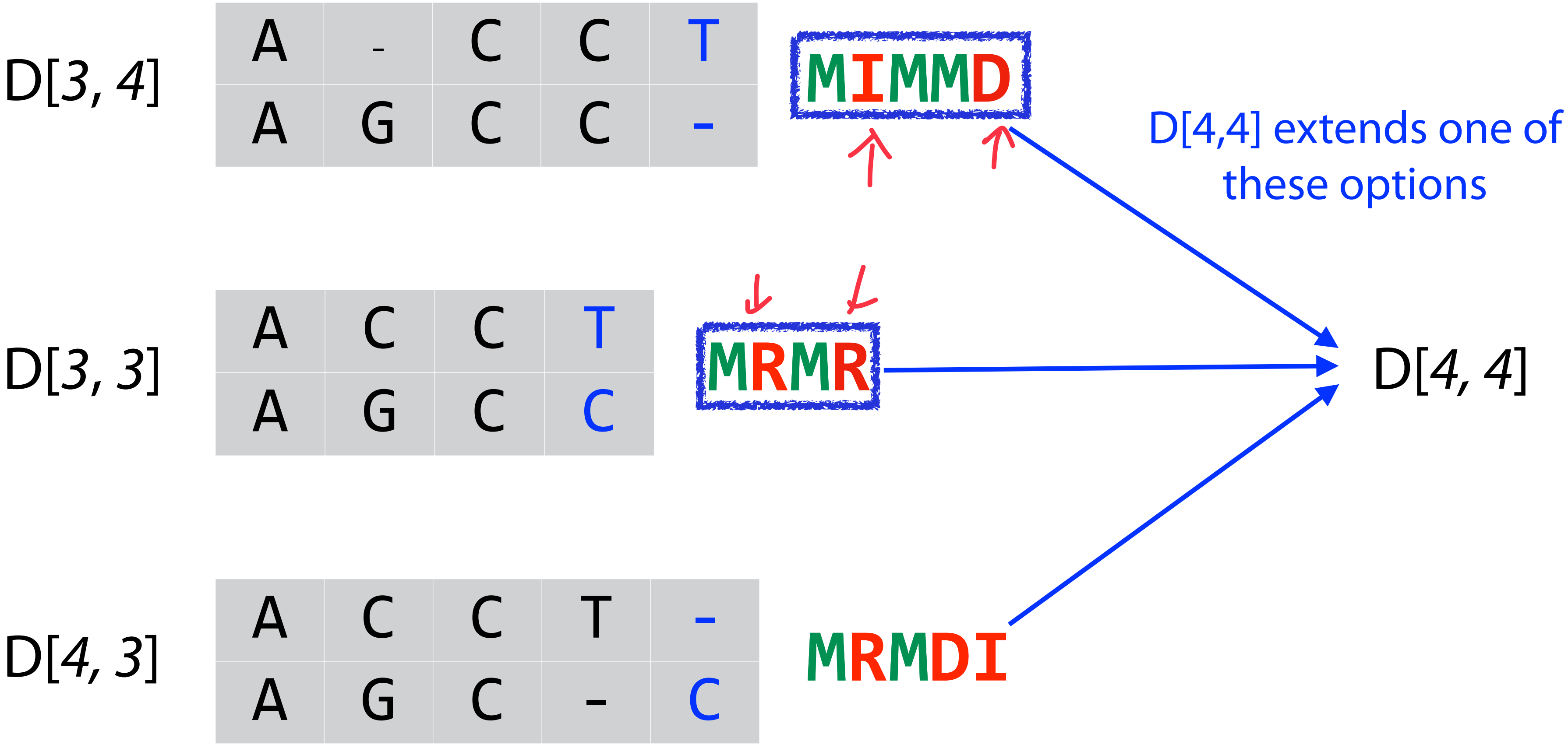
D[4, 4]

D[4,4] extends one of these options

Edit Distance

X: ACCT

Y: AGCC



D[3, 4]

A	-	C	C	T
A	G	C	C	-

MIMMD

D[3, 3]

A	C	C	T
A	G	C	C

MRMR

D[4, 3]

A	C	C	T	-
A	G	C	-	C

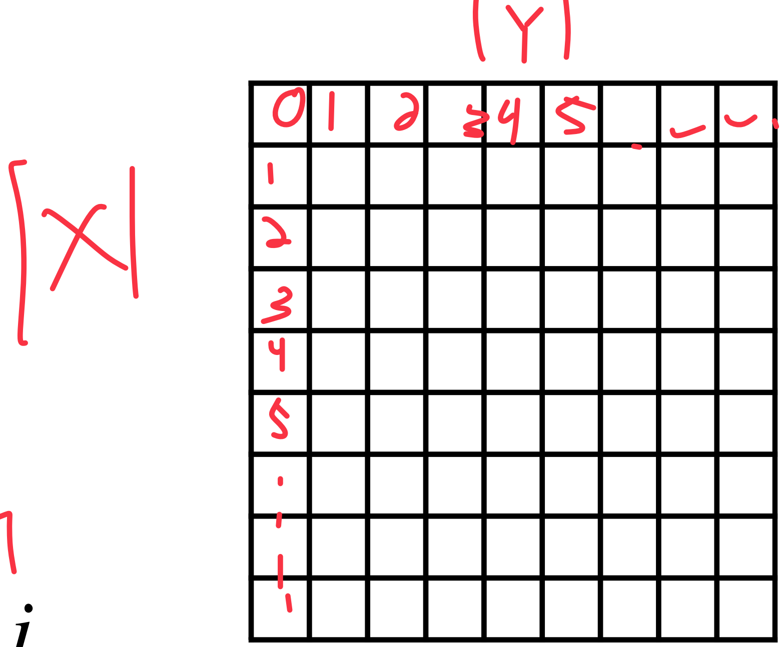
MRMDI

D[4,4] extends one of these options

D[4, 4]

Edit Distance

We can store D as a 2D matrix:

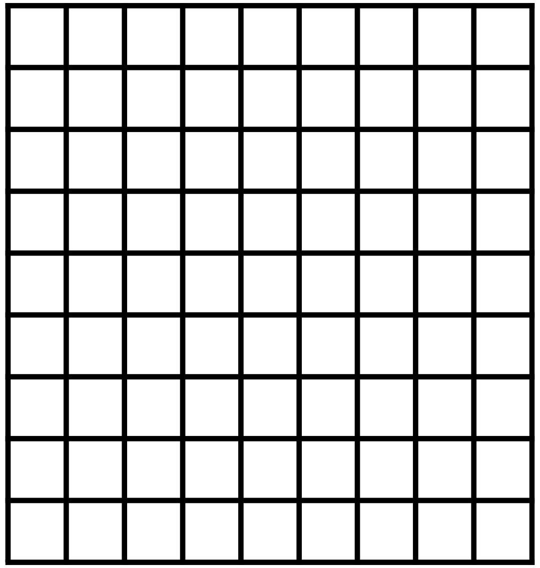


↓ D

Let $\overbrace{D[\underset{\uparrow}{0}, j] = j}$, and let $\overbrace{D[i, 0] = i}$

Edit Distance

We can store D as a 2D matrix:



Let $\overbrace{D[0,j] = j}^{\texttt{I}\text{s at beginning}}$, and let $\overbrace{D[i,0] = i}^{\texttt{D}\text{s at beginning}}$

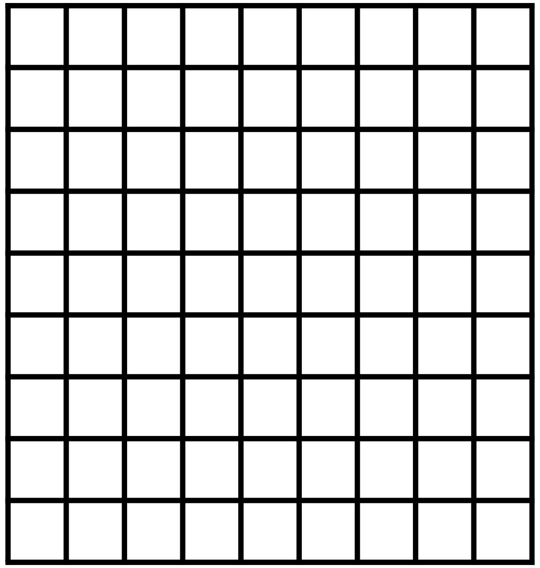
I

```
----BBBBBBB
    |||||||
AAAABBBBBBB
```

D

```
the longest----
    |||||||
----longest day
```

Edit Distance



We can store D as a 2D matrix:

Is at beginning **D**s at beginning

Let $D[0, j] = j$, and let $D[i, 0] = i$

Otherwise, let
$$D[i,j] = \min \begin{cases} \underline{D[i-1,j]} + 1 \\ \underline{D[i,j-1]} + 1 \\ \underline{D[i-1,j-1]} + \delta(x[i-1], y[j-1]) \end{cases}$$

Add 1x no Y

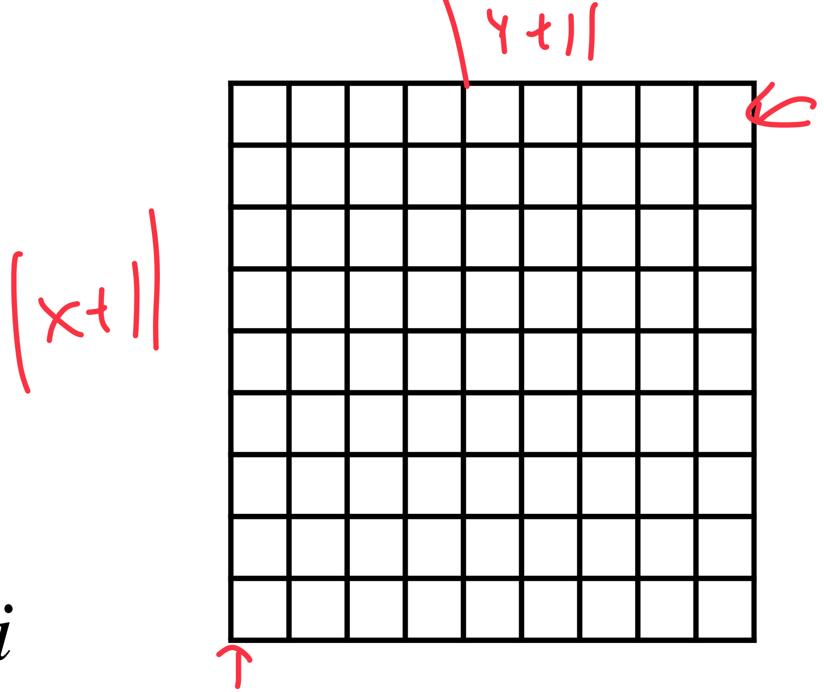
Add 0x 1Y

Add 1x,1Y

$\delta(a, b)$ is 0 if $a = b$, 1 otherwise

Edit Distance

We can store D as a 2D matrix:



Is at beginning

Ds at beginning

Let $D[0,j] = j$, and let $D[i,0] = i$

Otherwise, let $$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

vertical (**D**)

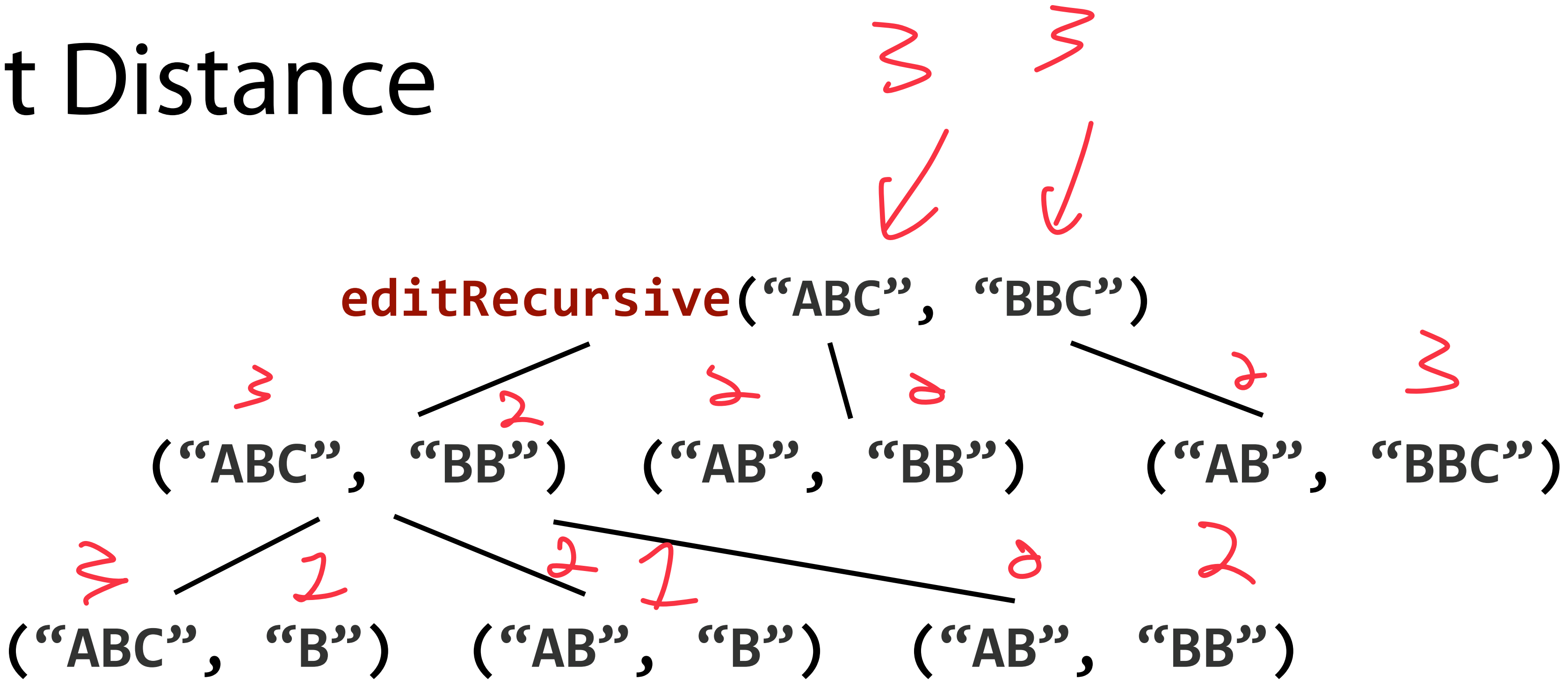
horizontal (**I**)

diagonal (**M** or **R**)

why not i, j?

$\delta(a,b)$ is 0 if $a = b$, 1 otherwise

Edit Distance



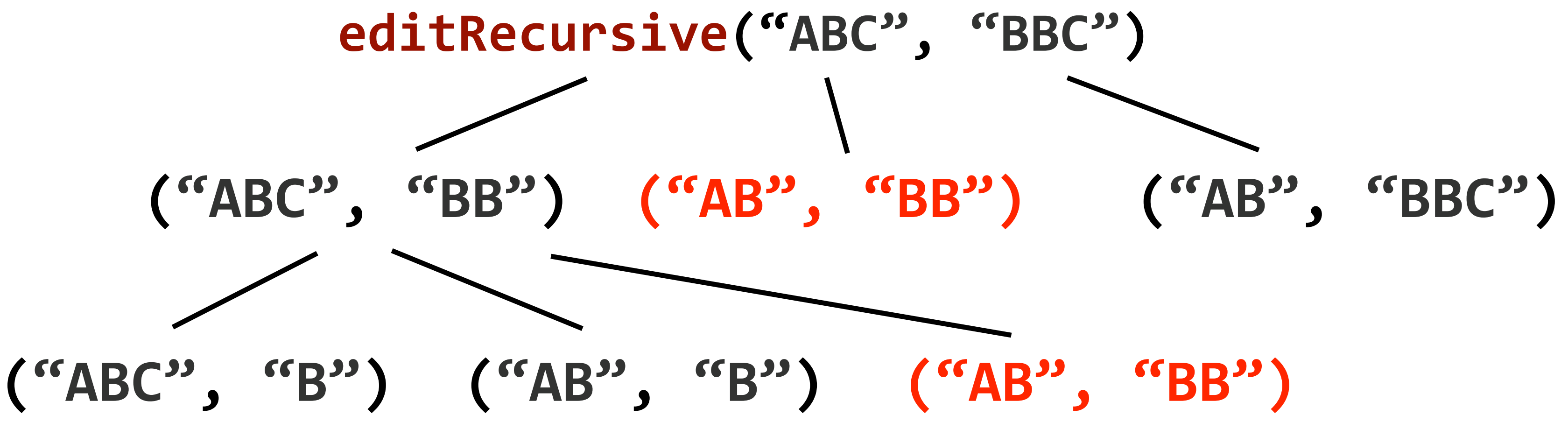
`editRecursive`(“ABC”, “BBC”)

(“ABC”, “BB”) (“AB”, “BB”) (“AB”, “BBC”)

(“ABC”, “B”) (“AB”, “B”) (“AB”, “BB”)

(only part of recursion tree shown)

Edit Distance



`editRecursive(“ABC”, “BBC”)`

(“ABC”, “BB”) (“AB”, “BB”) (“AB”, “BBC”)

(“ABC”, “B”) (“AB”, “B”) (“AB”, “BB”)

n(|x|+|y|)

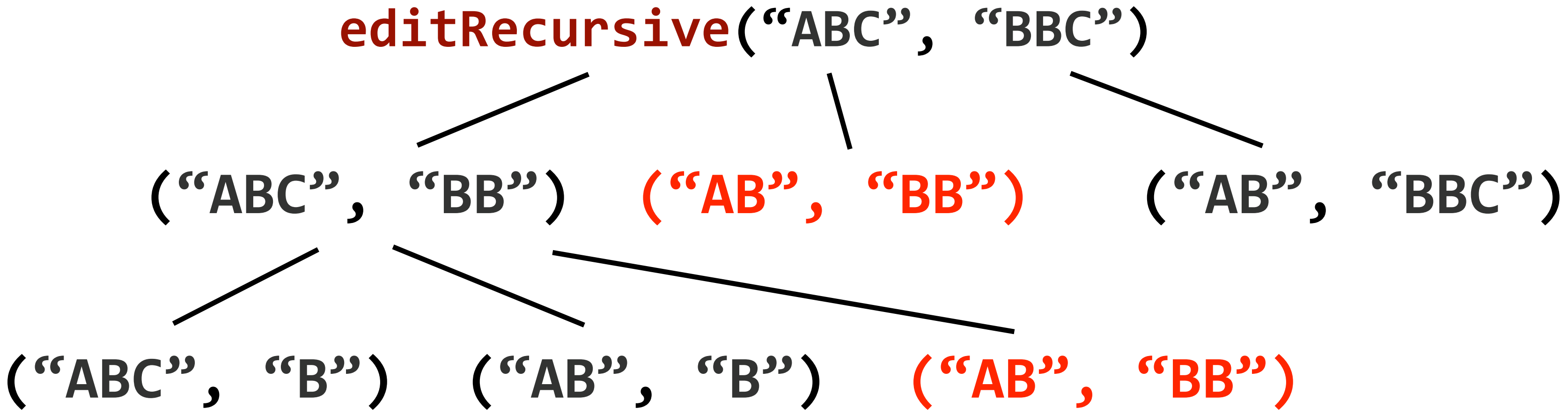
(only part of recursion tree shown)

`editRecursive(“Shakespeare”, “shake spear”)`

Calculate (“Shake”, “shake”) 8989 times

How can we address this problem?

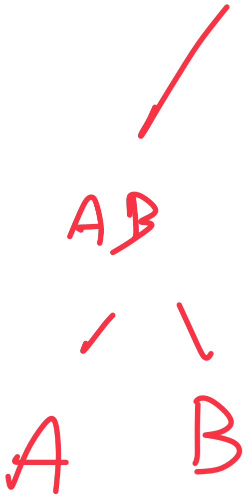
Edit Distance



editRecursive(“ABC”, “BBC”)

(“ABC”, “BB”) (“AB”, “BB”) (“AB”, “BBC”)

(“ABC”, “B”) (“AB”, “B”) (“AB”, “BB”)



Memoization: Top-down

Dynamic Programming: Bottom-up

Both: Solve individual sub-problems once

Edit Distance: dynamic programming

є is empty string

y

D: x

	є	G	C	T	A	T	G	C	C	A	C	G	C
є													
G													
C													
G													
T													
A													
T													
G													
C													
A													
C													
G													
C													

Let $n = |x|$, $m = |y|$

D: $\underline{(n+1)}$ x $\underline{(m+1)}$ matrix

D[i, j] = edit distance b/t length-i prefix of x and length-j prefix of y

Edit Distance: dynamic programming

Join Code: 225

Let $D[0, j] = j$, and let $D[i, 0] = i$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	A										
G													
C													
G													
T													
A													
T													
G													
C													
A													
C													
G													
C													

What is A?

D[*i*, *j*] = edit distance b/t length-*i* prefix of *x* and length-*j* prefix of *y*

Edit Distance: dynamic programming

Let $D[0, j] = j$, and let $D[i, 0] = i$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ			2										
G													
C													
G													
T													
A													
T													
G													
C													
A													
C													
G													
C													

What is A?

Edit Distance: dynamic programming

Let $D[0, j] = j$, and let $D[i, 0] = i$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1												
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

Edit Distance: dynamic programming

Let $D[0, j] = j$, and let $D[i, 0] = i$

Y

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1												
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

X

- - -
G C T

Edit Distance: dynamic programming

Let $D[0, j] = j$, and let $D[i, 0] = i$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1												
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

```
GCGTAT

------
```

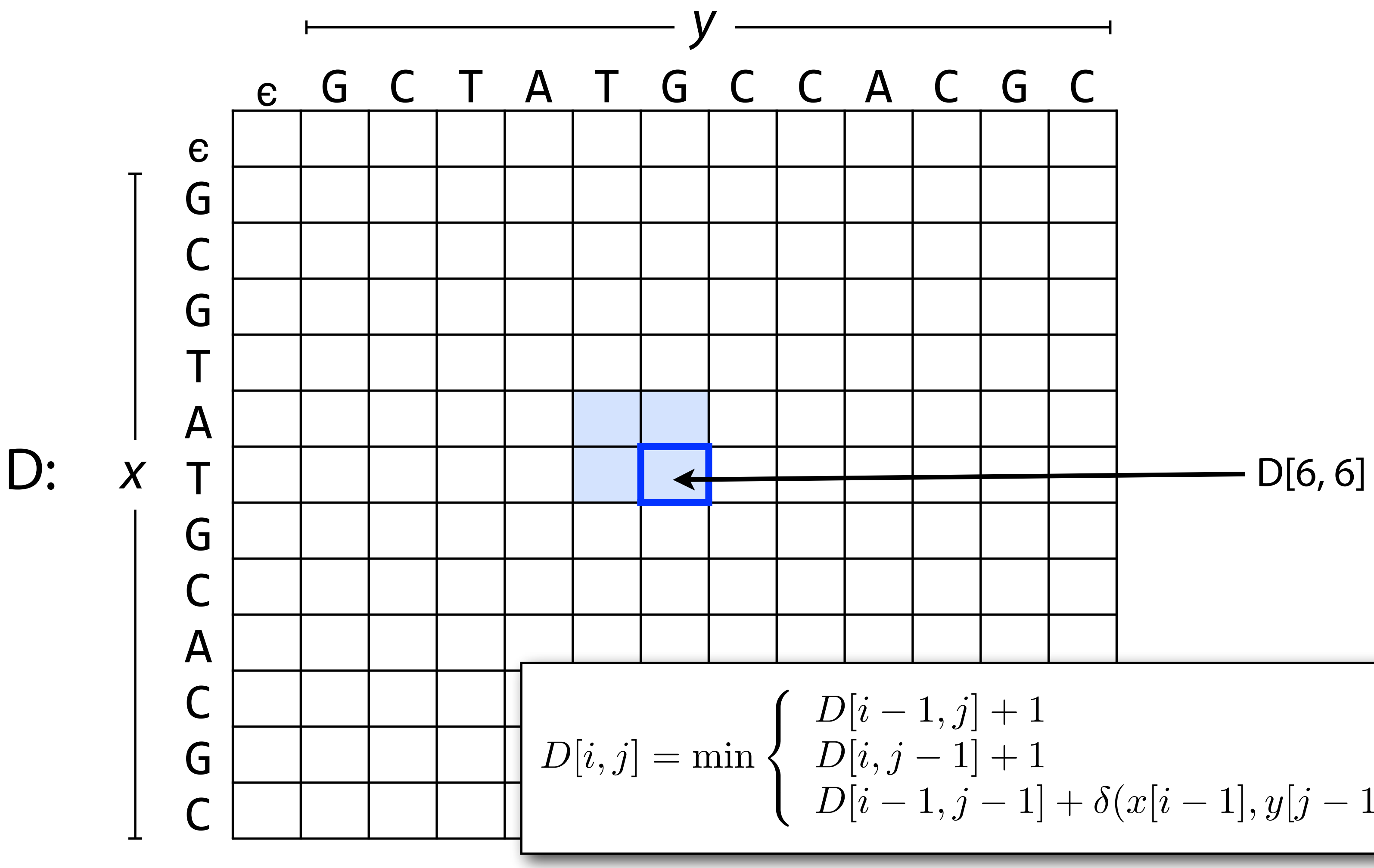
Edit Distance: dynamic programming

Let $D[0, j] = j$, and let $D[i, 0] = i$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1												
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

D is one row / column larger than x / y!

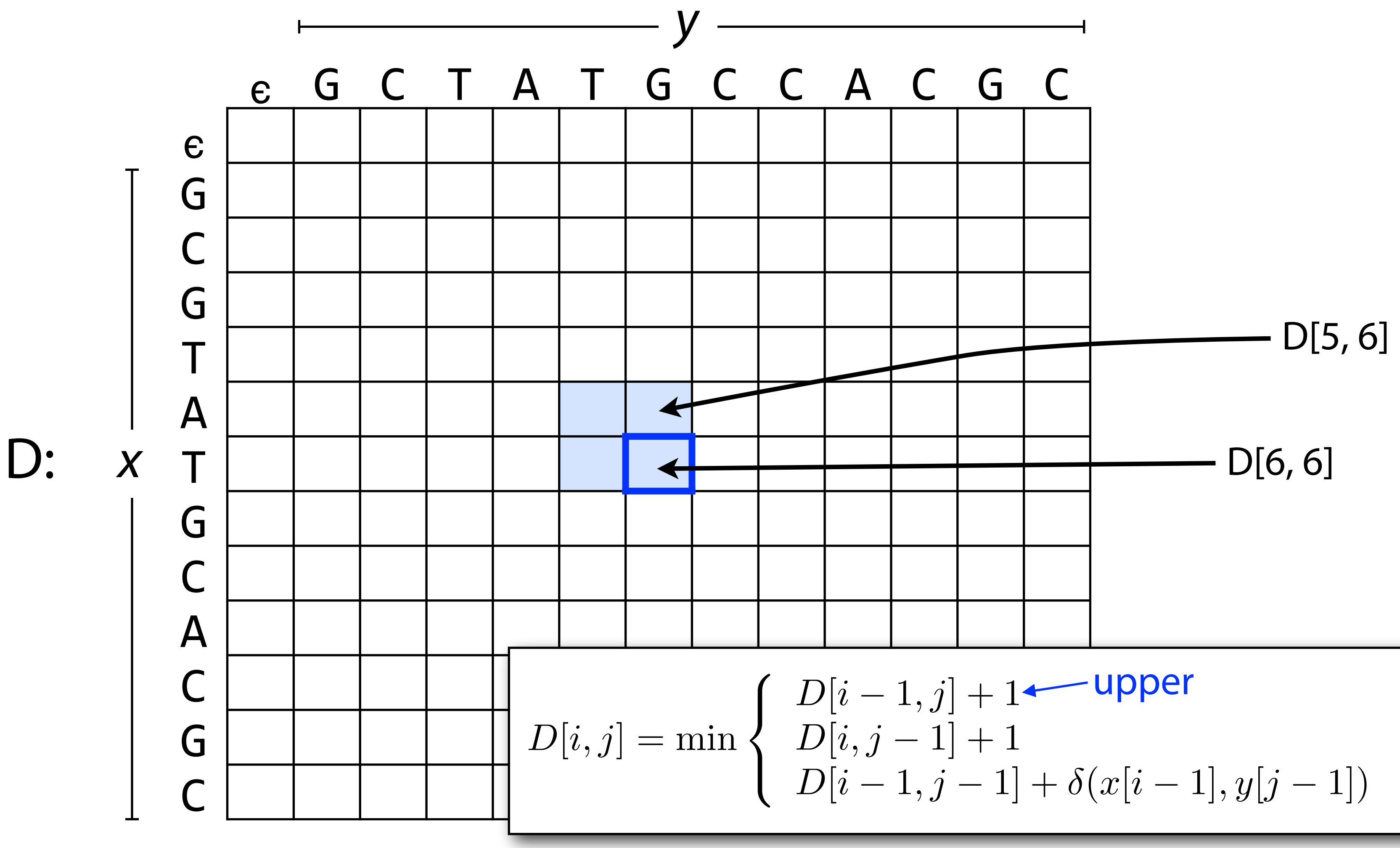
Edit Distance: dynamic programming



$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

Cell depends upon its upper, left, and upper-left neighbors

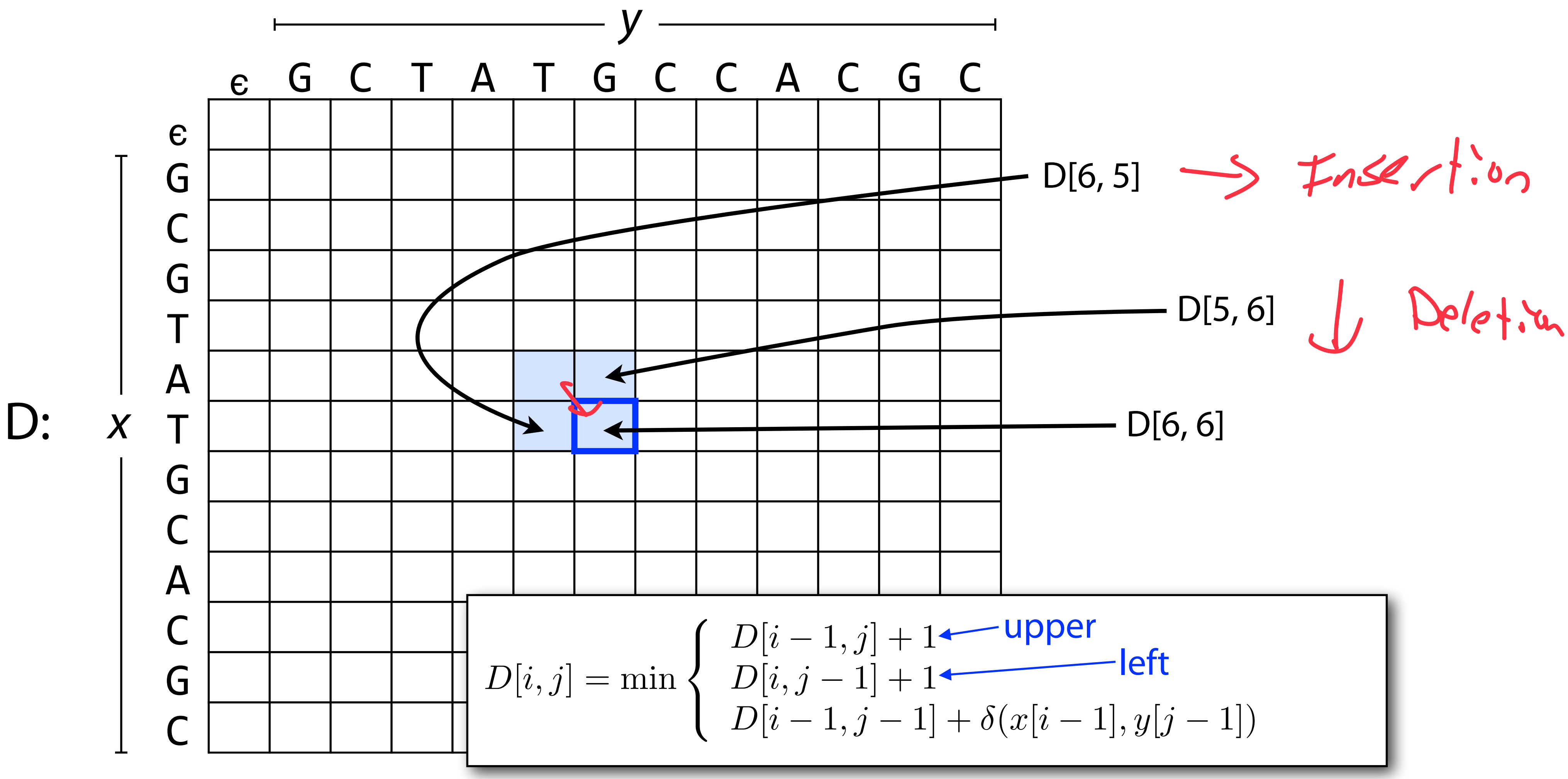
Edit Distance: dynamic programming



$$D[i,j] = \min \begin{cases} D[i-1,j] + 1 \leftarrow \text{upper} \\ D[i,j-1] + 1 \\ D[i-1,j-1] + \delta(x[i-1], y[j-1]) \end{cases}$$

Cell depends upon its upper, left, and upper-left neighbors

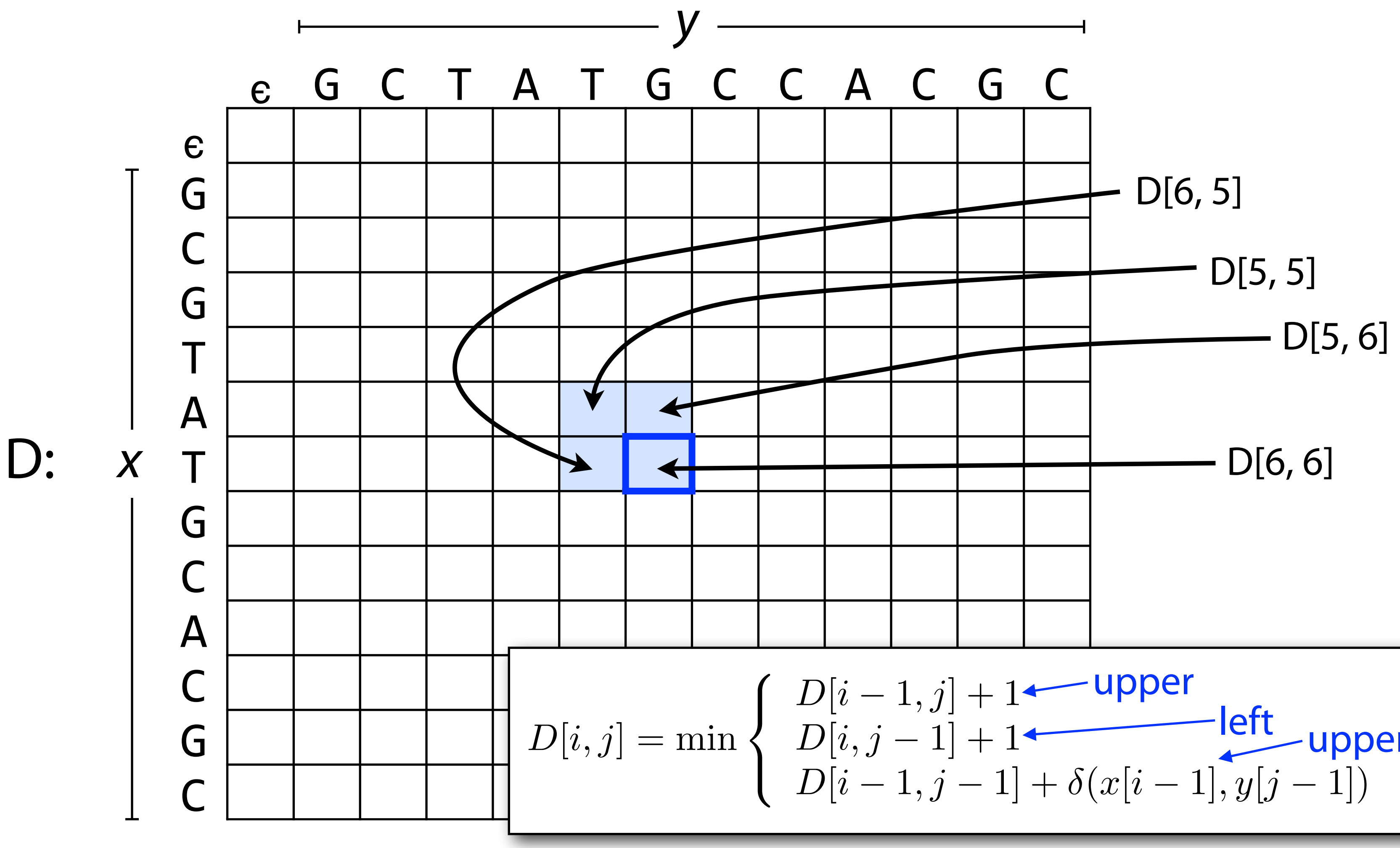
Edit Distance: dynamic programming



$$D[i,j] = \min \begin{cases} D[i-1,j] + 1 \leftarrow \text{upper} \\ D[i,j-1] + 1 \leftarrow \text{left} \\ D[i-1,j-1] + \delta(x[i-1], y[j-1]) \end{cases}$$

Cell depends upon its upper, left, and upper-left neighbors

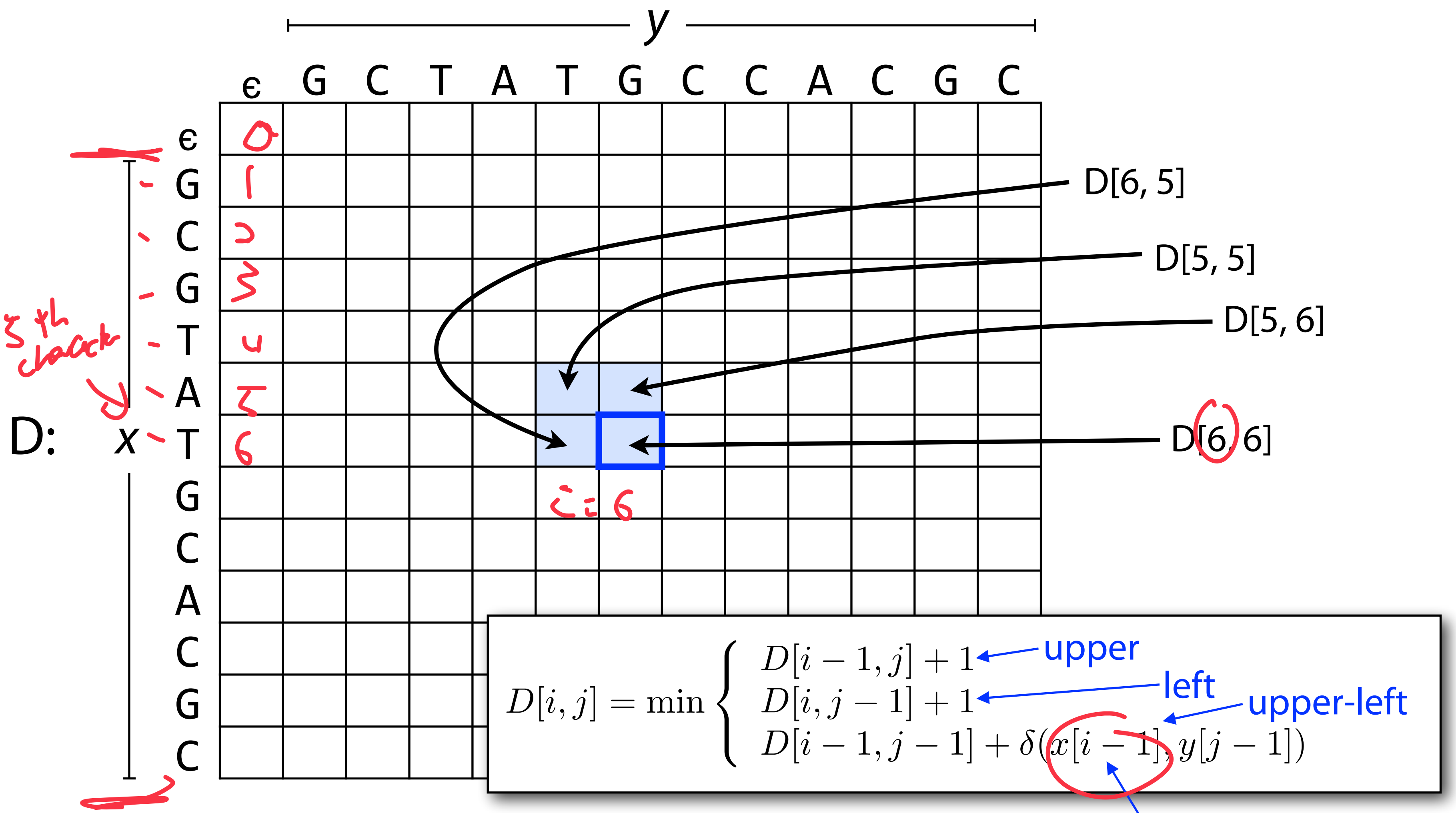
Edit Distance: dynamic programming



$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \leftarrow \text{upper} \\ D[i,j-1]+1 \leftarrow \text{left} \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \quad \text{upper-left} \end{cases}$$

Cell depends upon its upper, left, and upper-left neighbors

Edit Distance: dynamic programming



$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

upper

left

upper-left

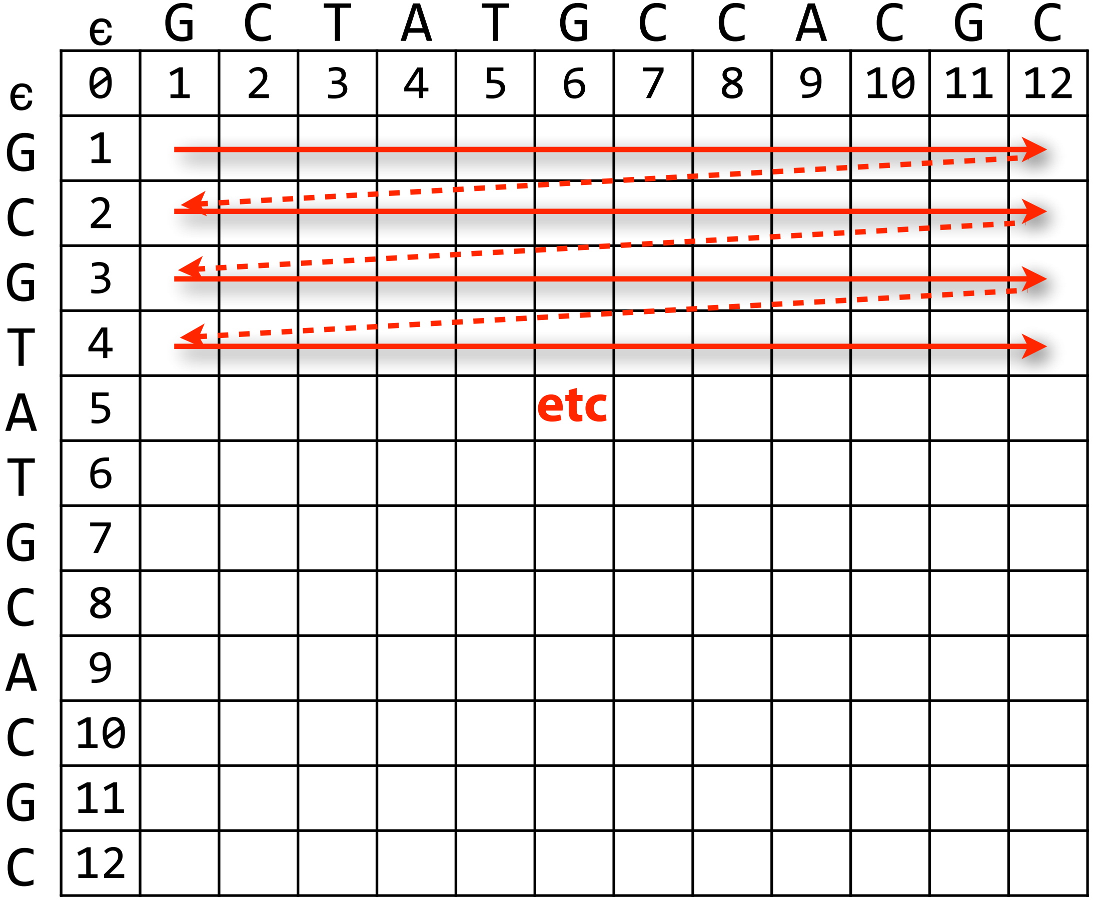
Cell depends upon its upper, left, and upper-left neighbors

i, j are D-based

D[0] is 'empty string'

Edit Distance: dynamic programming

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$



	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1												
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

Fill remaining cells from top row to bottom and from left to right

Edit Distance: dynamic programming

Join Code: 225

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

	ε	G	C	T	A	T	G	C	C	A	C	G	C
ε	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	?											
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

What goes here in **`i=1,j=1`**?

$$\begin{array}{c} G \\ G \end{array}$$

$$\begin{array}{cc} - & G \\ G & - \end{array}$$

$$\begin{array}{cc} G & - \\ - & G \end{array}$$

Edit Distance: dynamic programming

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	?											
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

What goes here in **`i=1,j=1`**?

`x[i-1]` = ‘**`G`**’,
`y[j-1]` = ‘**`G`**’,
so `delt = 0`

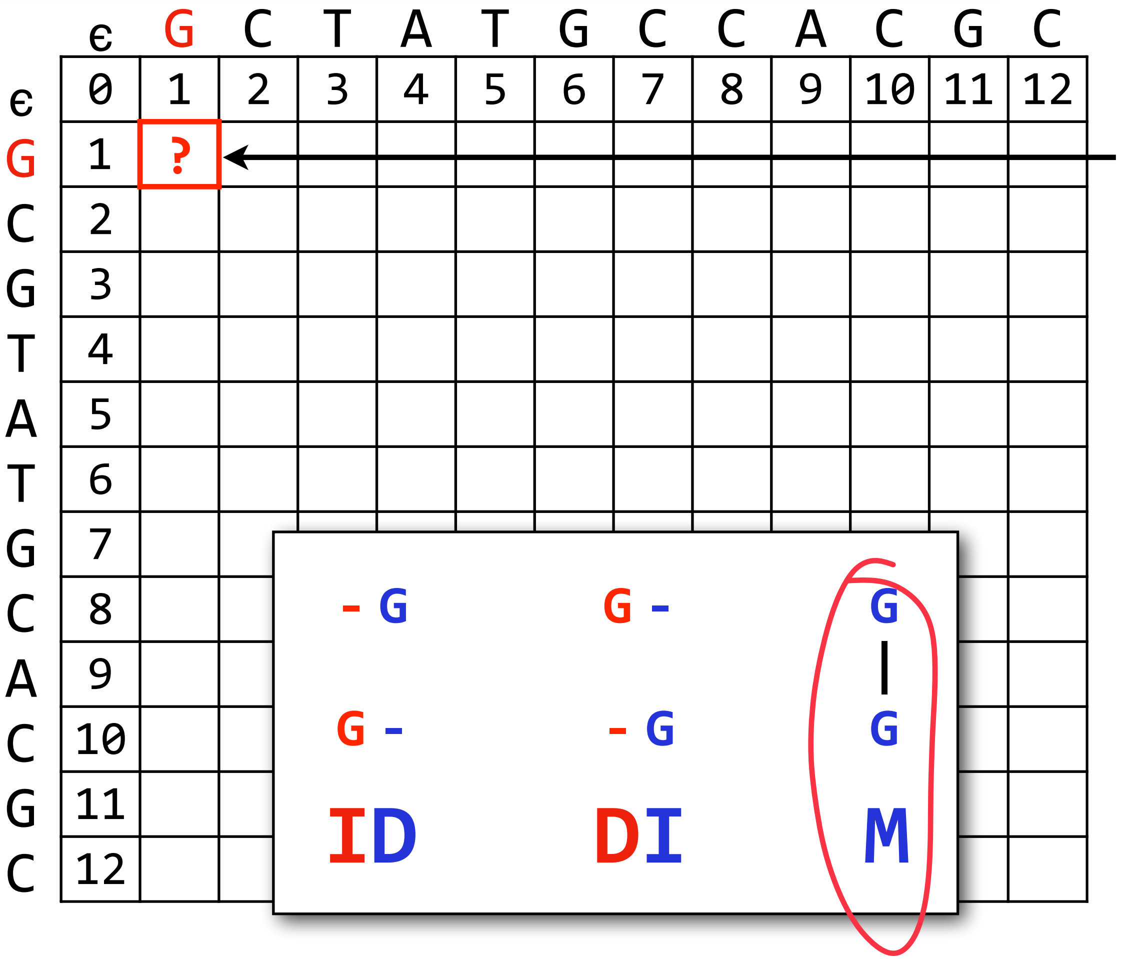
- G
G -
ID

G -
- G
DI

G
|
G
M

Edit Distance: dynamic programming

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$



	ε	G	C	T	A	T	G	C	C	A	C	G	C
ε	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	?											
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

What goes here in **`i=1,j=1`**?

`x[i-1]` = '**G**',
`y[j-1]` = '**G**',
so **`delt`** = `0`

```
D[i, j] = min(D[i-1, j]+1,
              D[i, j-1]+1,
              D[i-1, j-1]+delt)
        = min(1 + 1, 1 + 1, 0 + 0)
        = 0
```

Edit Distance: dynamic programming

Join Code: 225

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	?										
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

What goes here in **i=1,j=2**?

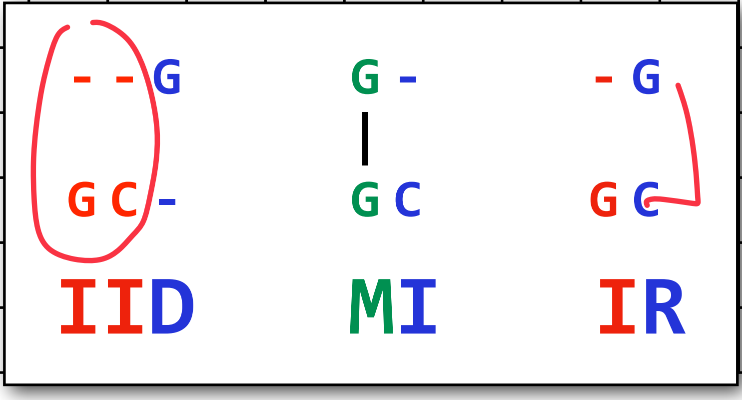
Edit Distance: dynamic programming

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	?										
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

What goes here in **i=1,j=2**?

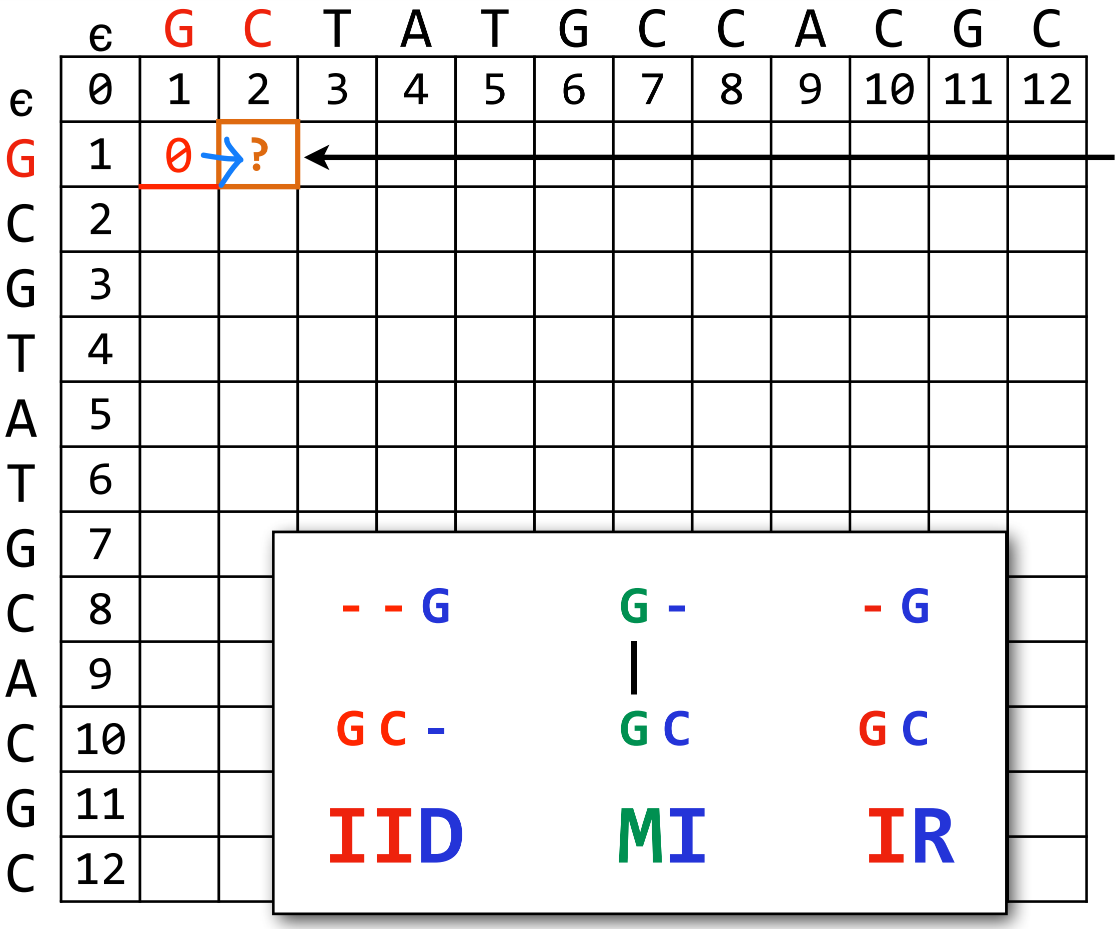
x[i-1] = '**G**',
y[j-1] = '**C**',
so **delt = 1**



```
- - G        G -        - G
             |
G C -        G C        G C
IID          MI         IR
```

Edit Distance: dynamic programming

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$



	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	?										
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

- - G	G -	- G
	|	
G C -	G C	G C
IID	**MI**	**IR**

What goes here in **`i=1,j=2`**?

`x[i-1]` = ‘`G`’,
`y[j-1]` = ‘`C`’,
so `delt` = `1`

```
D[i, j] = min(D[i-1, j]+1,
              D[i, j-1]+1,
              D[i-1, j-1]+delt)
        = min(2 + 1, 0 + 1, 1 + 1)
        = 1
```

Edit Distance: dynamic programming

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6					
C	2	1	0	1	2	3	4	5					
G	3	2	1	1	2	3	3	4					
T	4	3	2	1	2	2	3	?					
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

What goes here in **i=4,j=7**?

+1

3+1

C y

Diagonal

3+1 or 3+0

+1 b/c $\frac{C}{T}$ doesn't match

Edit Distance: dynamic programming

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

	ε	G	C	T	A	T	G	C	C	A	C	G	C
ε	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6					
C	2	1	0	1	2	3	4	5					
G	3	2	1	1	2	3	3	4					
T	4	3	2	1	2	2	3	4					
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

What goes here in **i=4,j=7**?

x[i-1] = '**T**',
y[j-1] = '**C**',
so **delt** = **1**

```
GC---GT        GC-GT--

GCTATGC        GCTATGC

MMIIIMR        MMIRMII
```

```
D[i, j] = min(D[i-1, j]+1,
              D[i, j-1]+1,
              D[i-1, j-1]+delt)
        = min(4 + 1, 3 + 1, 3 + 1)
        = 4
```

Edit Distance: dynamic programming

$$D[i,j] = \min \begin{cases} D[i-1,j]+1 \\ D[i,j-1]+1 \\ D[i-1,j-1]+\delta(x[i-1],y[j-1]) \end{cases}$$

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

Fill remaining cells from top row to bottom and from left to right

Edit distance for x, y

Edit Distance

Y

X

	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	~~A~~ 2	2	3
A	2	1	0	~~B~~ 1	
T	3	2	~~C~~ 2		

Edit Distance

Y

X		ε	C	A	A	T
	ε	0	1	2	3	4
	C	1	0	**1**	2	3
	A	2	1	0	**B**	
	T	3	2	**C**		

Edit Distance

Y

X

	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	
T	3	2	C		

Edit Distance

Y

		ϵ	C	A	A	T
	ϵ	0	1	2	3	4
X	C	1	0	**1**	2	3
	A	2	1	0	**1**	
	T	3	2	**1**		

Edit Distance

Y

X

	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T	3	2	1	1	1

`eMatrix buildEditMatrix(X, Y)`

Input:

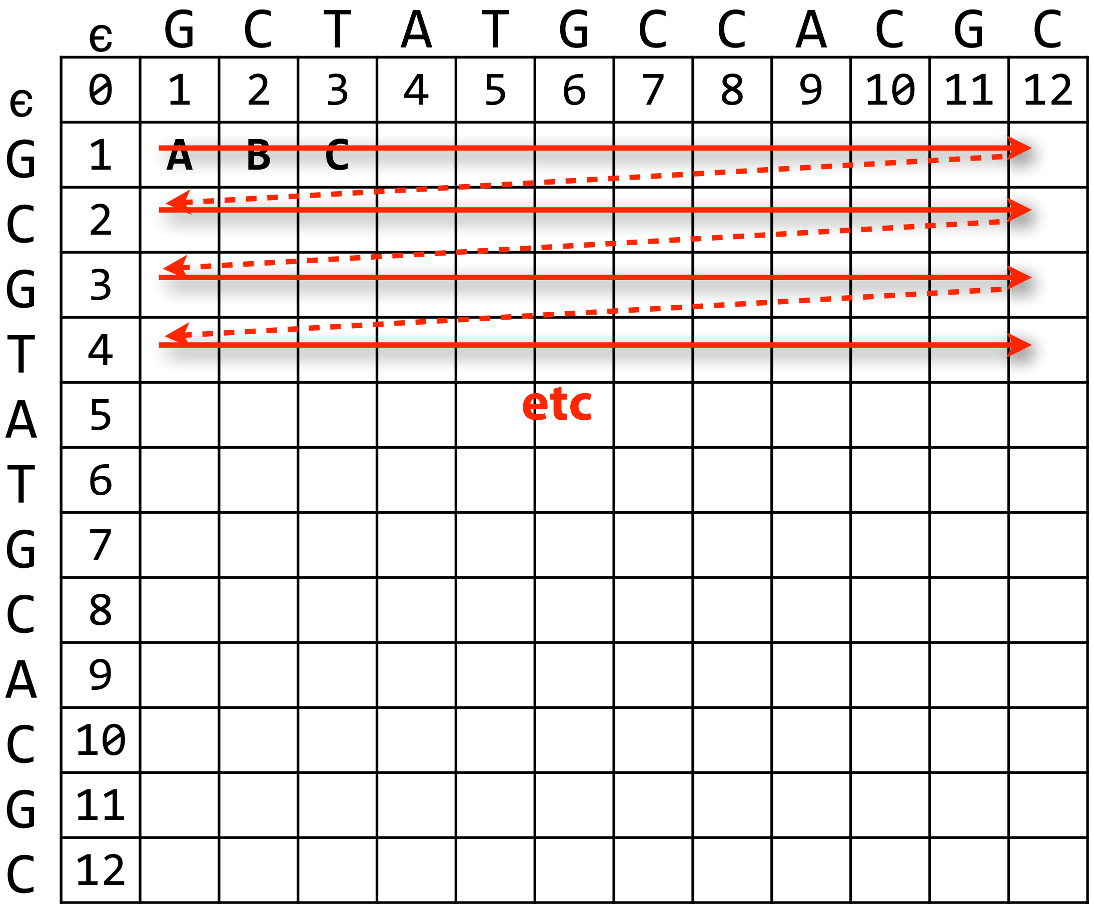
- **`string X`**: Input string X
- **`string Y`**: Input string Y

Output:

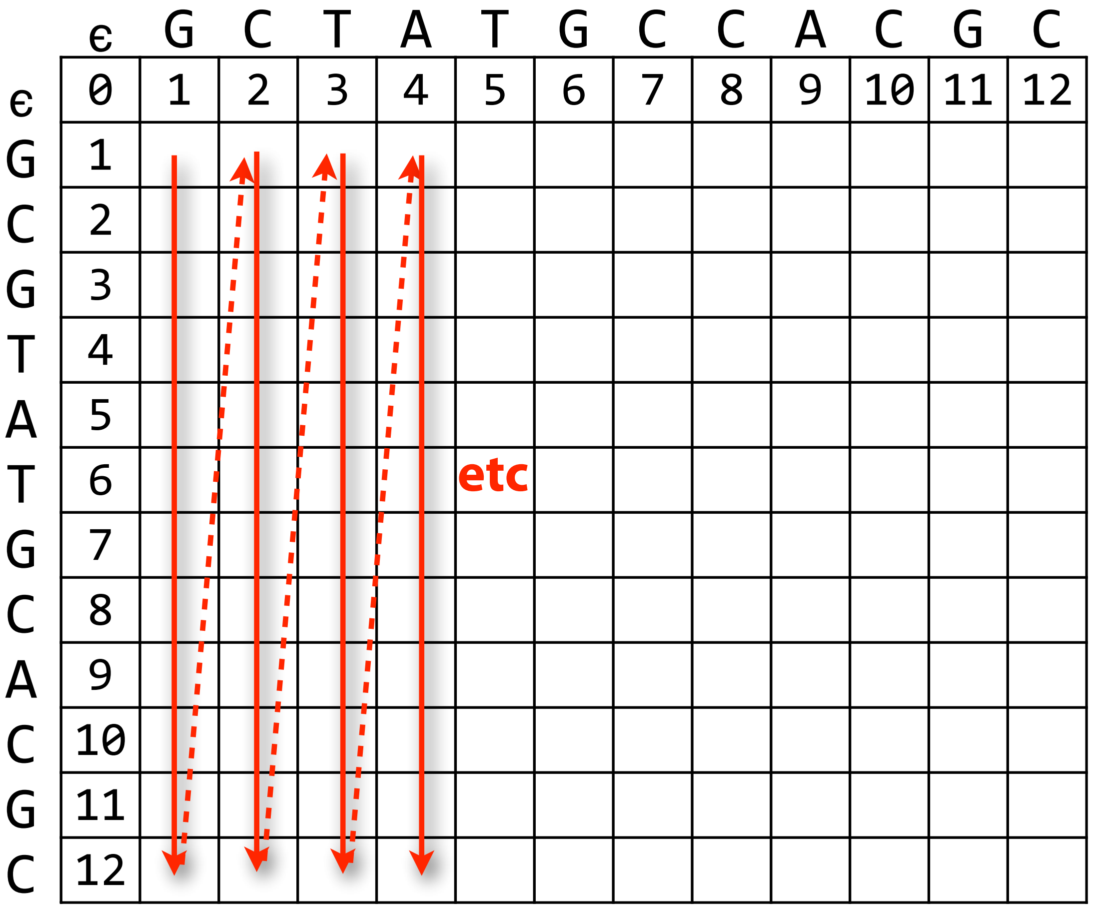
- `eMatrix` : vector<vector<int>> storing all optimal edit distances

	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T	3	2	1	1	1

Edit Distance: dynamic programming

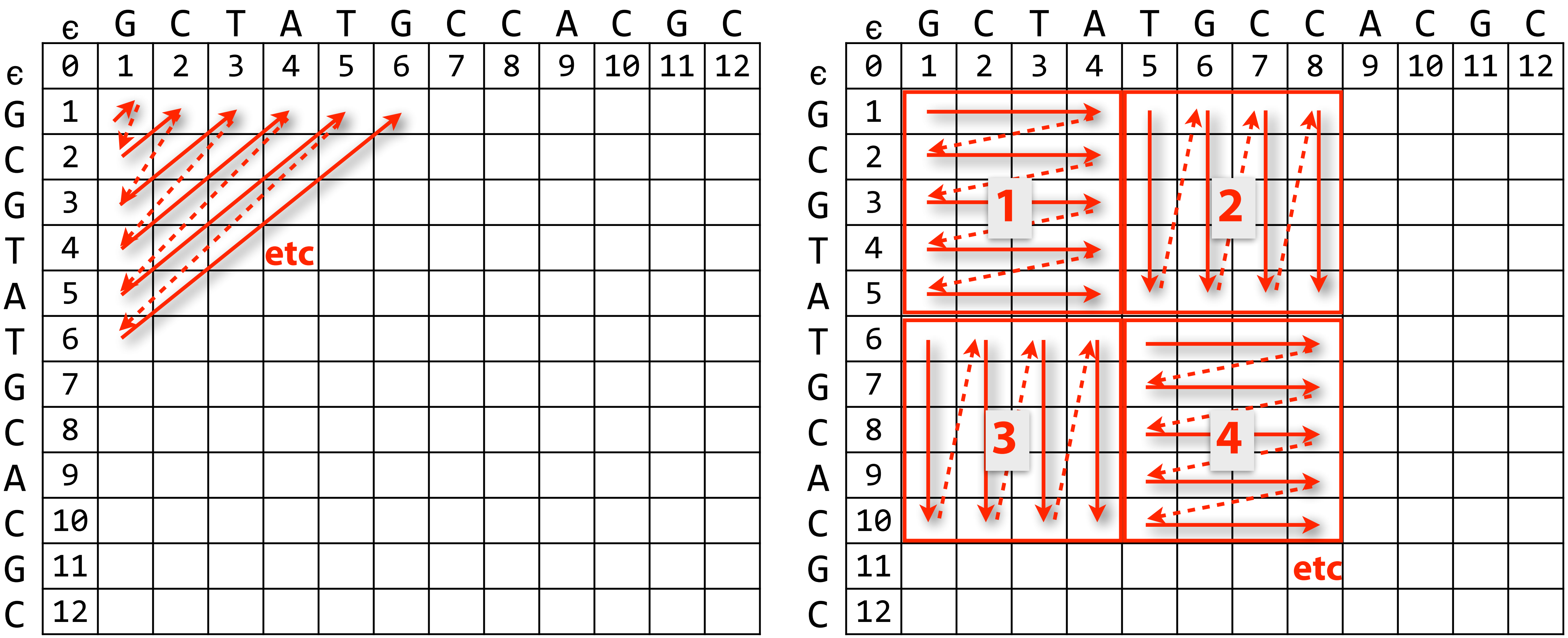


	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	A	B	C									
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

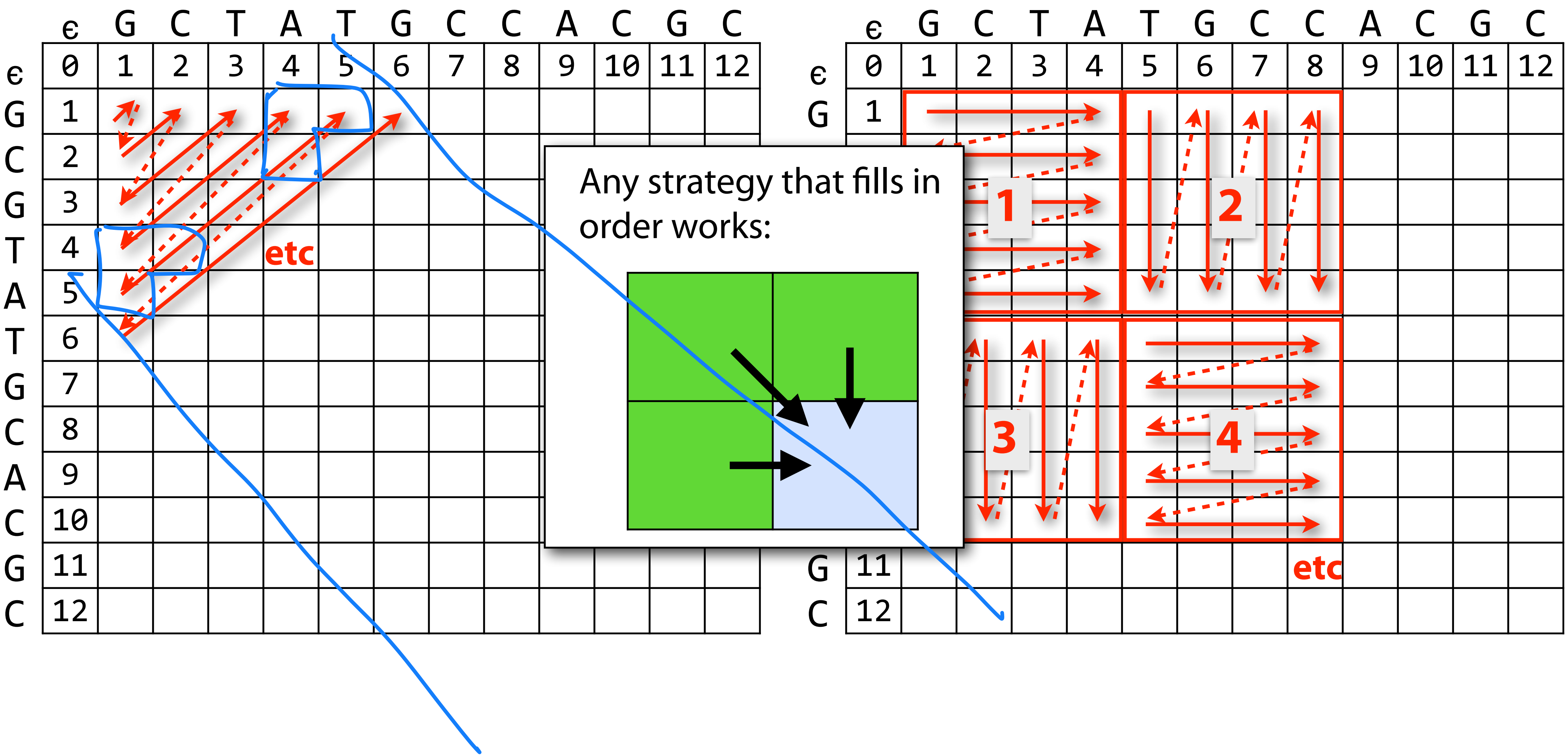


	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1												
C	2												
G	3												
T	4												
A	5												
T	6												
G	7												
C	8												
A	9												
C	10												
G	11												
C	12												

Edit Distance: dynamic programming



Edit Distance: dynamic programming



Any strategy that fills in order works:

Assignment 11: a_edist

Learning Objective:

Use dynamic programming to build an edit distance matrix

Construct an optimal edit string from the edit matrix

Consider: Does substitution, insertion, and deletion need to have the same 'weight' as a penalty? How could you modify the code to take a user-specified input for each?

Edit Distance

	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T	3	2	1	1	1

Edit distance for x, y

But where and what is the edit?

Edit Distance

Traceback corresponds to an optimal alignment / edit transcript

At each step, ask: which neighbor (⇖, ⇐ or ⇑) gave the minimum?

	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T	3	2	1	1	**1**

1+0

2+1 X

1+1 x

← Q: How did I get here?

Edit Distance

Traceback corresponds to an optimal alignment / edit transcript

At each step, ask: which neighbor (⇖, ⇐ or ⇑) gave the minimum?

	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	**2**
T	3	2	1	1	1

$D[2,\ 4] = \underline{2 + 1 = 1?}$ ✗

Q: How did I get here?

Edit Distance

Traceback corresponds to an optimal alignment / edit transcript

At each step, ask: which neighbor (⇖, ⇐ or ⇑) gave the minimum?

	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T	3	2	1	1	1

$D[3,\ 3] =$ 1 + 1 = 1 ✗

← Q: How did I get here?

Edit Distance

Traceback corresponds to an optimal alignment / edit transcript

At each step, ask: which neighbor (⇖, ⇐ or ⇑) gave the minimum?

	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T	3	2	1	1	1

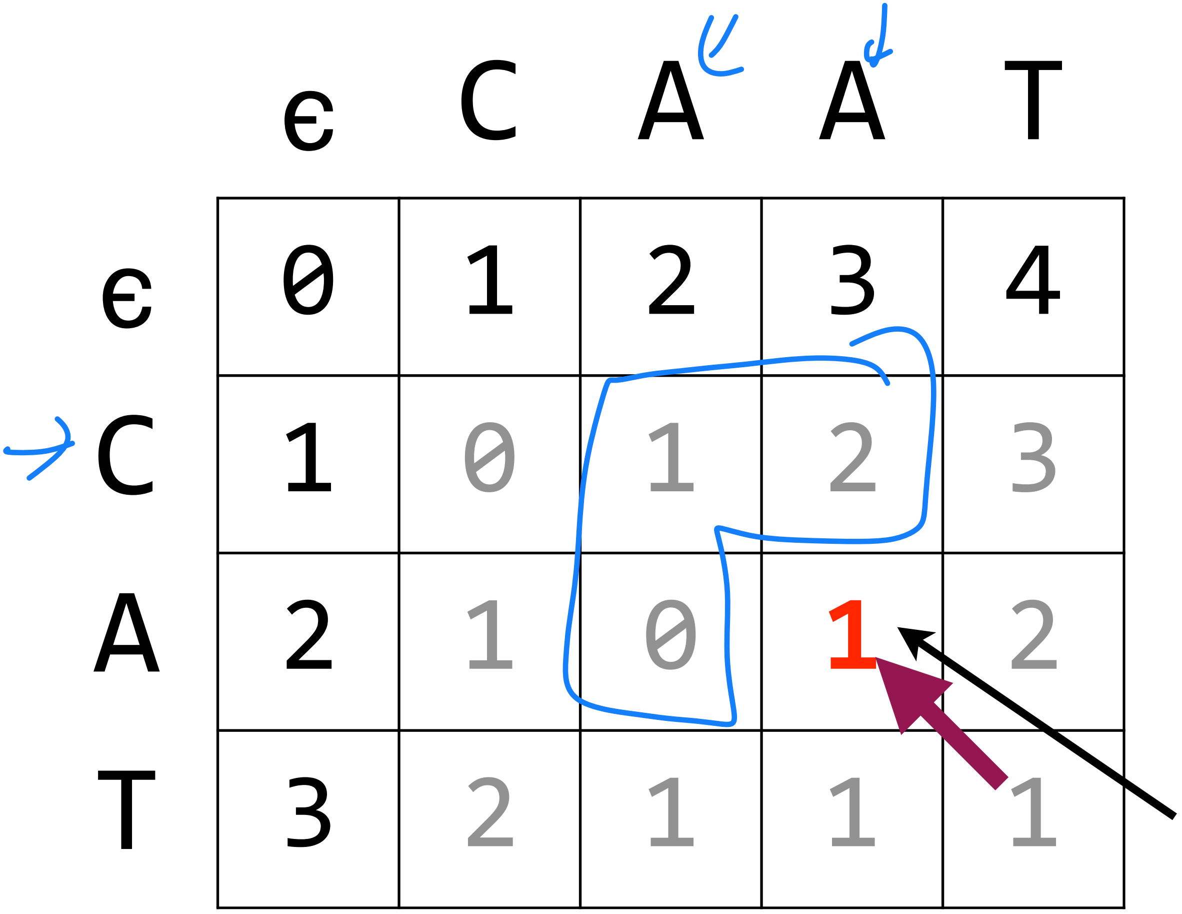
$D[2,\ 3] = \underline{1 + 0 = 1}$ ✓

Q: How did I get here?

Edit Distance

Traceback corresponds to an optimal alignment / edit transcript

At each step, ask: which neighbor (⇖, ⇐ or ⇑) gave the minimum?



	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T	3	2	1	1	1

Q: How did I get here?

$D[1,\ 3] =$ 2+1 ≠ 1

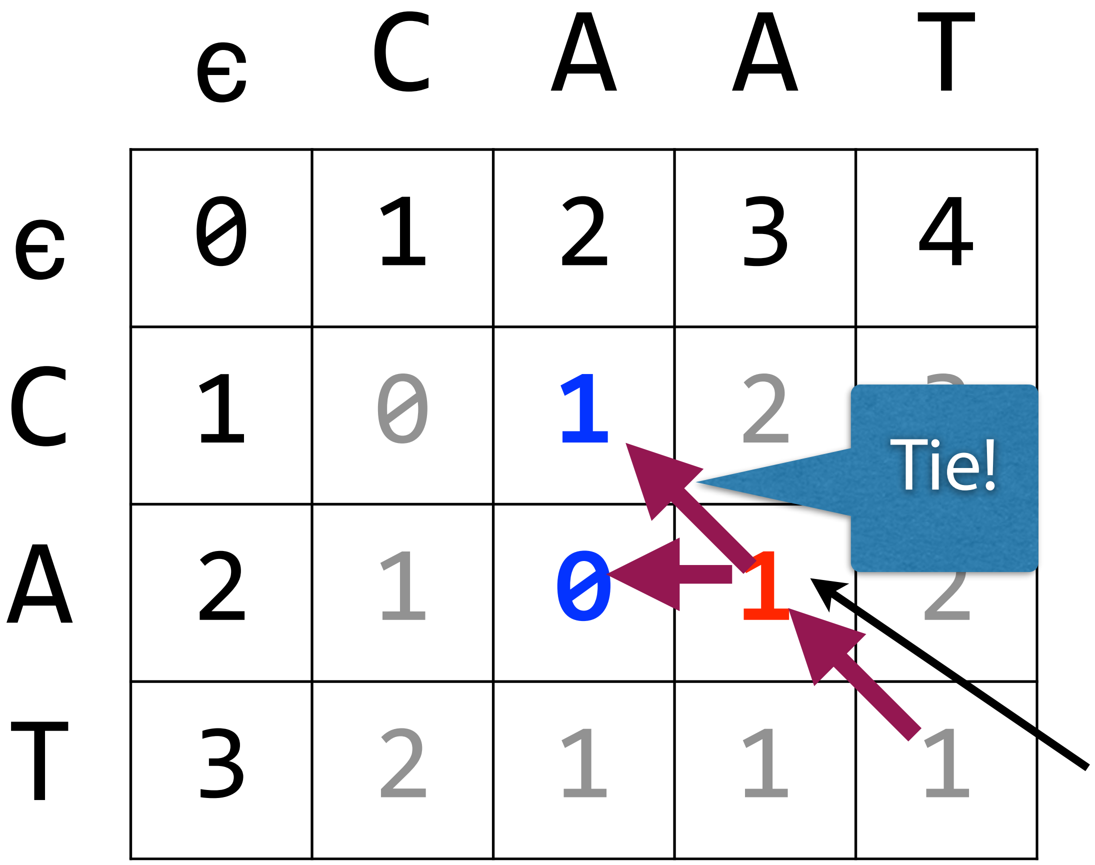
$D[1,\ 2] =$ 1+0 = 1

$D[2,\ 2] =$ 0+1 = 1

Edit Distance

Traceback corresponds to an optimal alignment / edit transcript

At each step, ask: which neighbor (⇖, ⇐ or ⇑) gave the minimum?



	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	[illegible]
A	2	1	0	1	2
T	3	2	1	1	1

Tie!

$$D[1,\ \ 3] = 2 + 1$$

$$D[1,\ \ 2] = 1 + 0$$

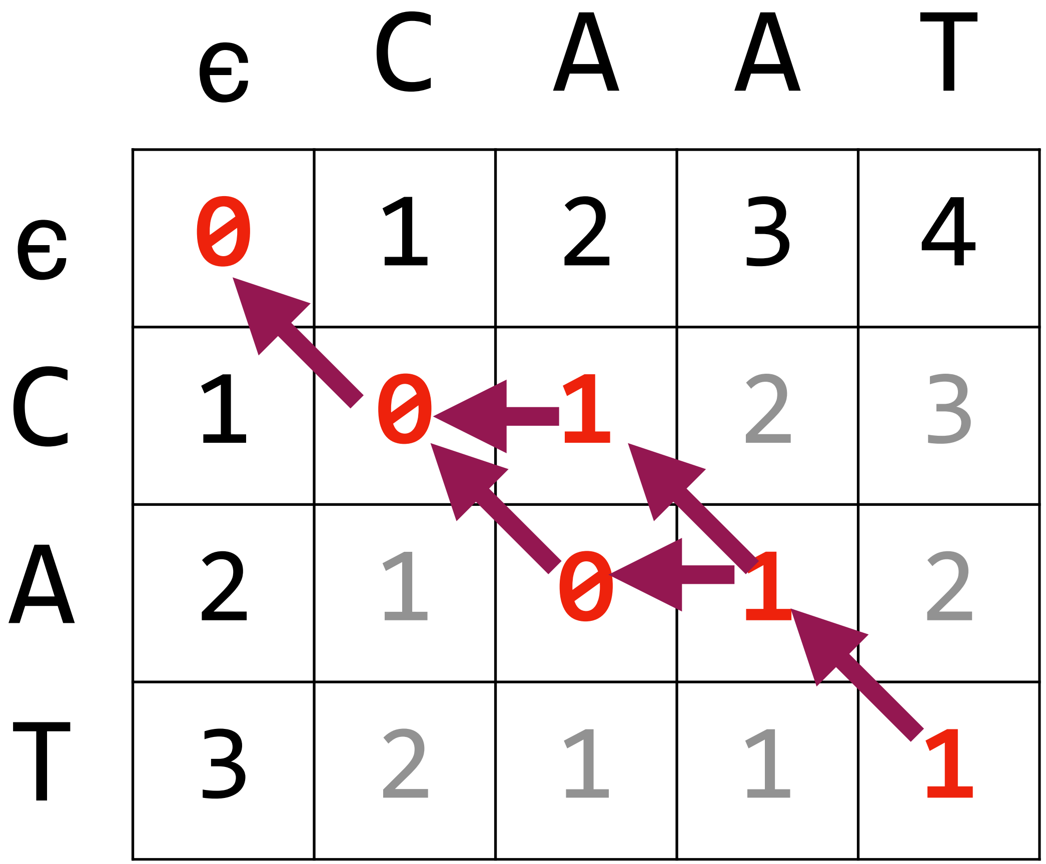
$$D[2,\ \ 2] = 0 + 1$$

Q: How did I get here?

Edit Distance

Traceback corresponds to an optimal alignment / edit transcript

At each step, ask: which neighbor (⇖, ⇐ or ⇑) gave the minimum?

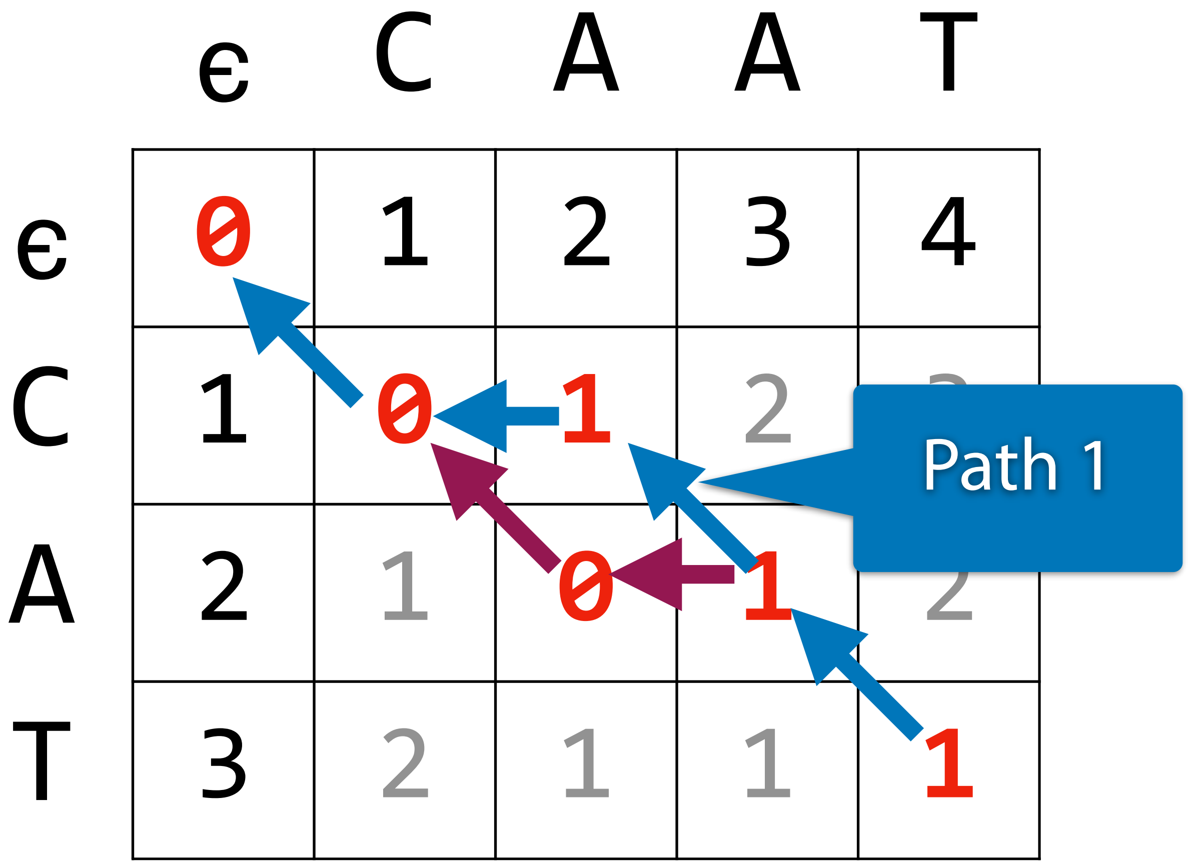


	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T	3	2	1	1	1

Edit Distance

Traceback corresponds to an optimal alignment / edit transcript

At each step, ask: which neighbor (⇖, ⇐ or ⇑) gave the minimum?



	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	[illegible]
A	2	1	0	1	2
T	3	2	1	1	1

Path 1

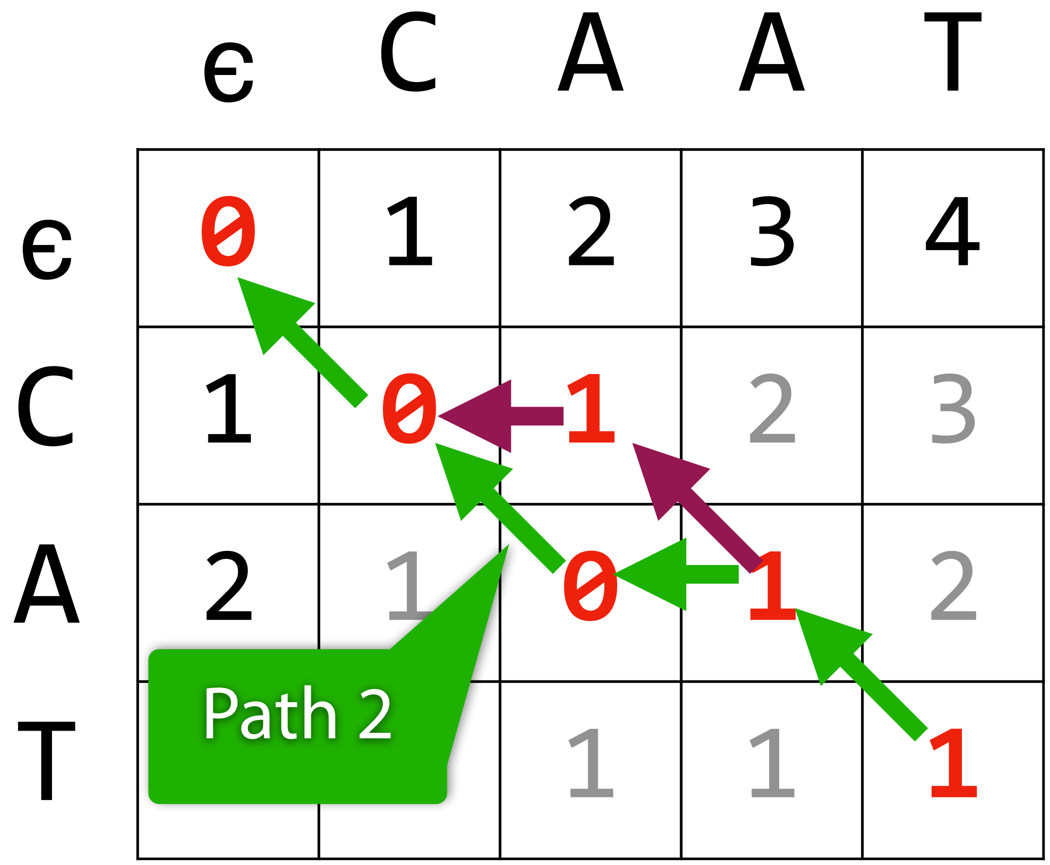
MIMM

```
C-AT
|  ||
CAAT
```

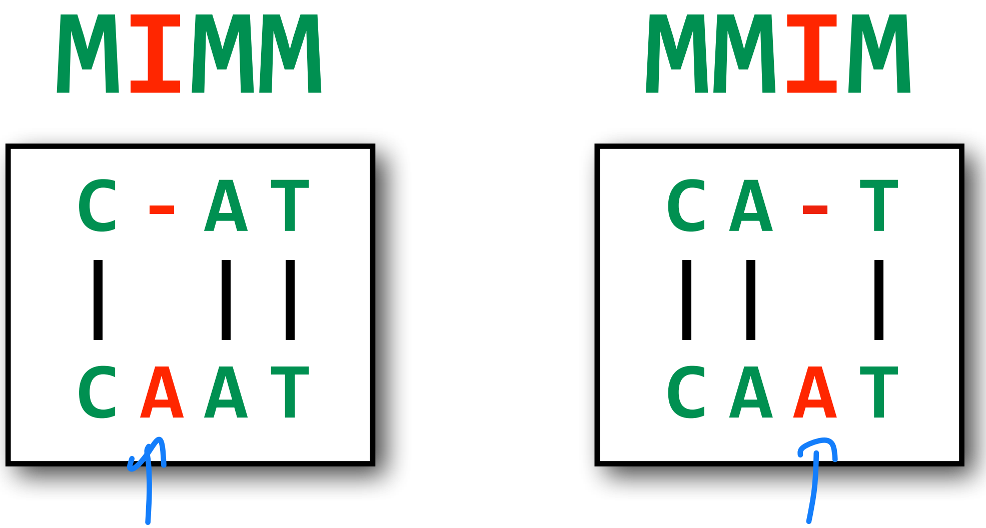
Edit Distance

Traceback corresponds to an optimal alignment / edit transcript

At each step, ask: which neighbor (⇖, ⇐ or ⇑) gave the minimum?



	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T			1	1	1



MIMM

```
C - A T
|   | |
C A A T
```

MMIM

```
C A - T
| |   |
C A A T
```

Edit Distance

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

Q: How did I get here?

Edit Distance

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

$$D[11,\quad 12] = 3 + 1$$

$$D[11,\quad 11] = 2 + 0$$

$$D[12,\quad 11] = 3 + 1$$

A: From here

Q: How did I get here?

Edit Distance

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

Q: How did I get here?

Edit Distance

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

$$D[10,\quad 11] = 3 + 1$$

$$D[10,\quad 10] = 2 + 0$$

$$D[11,\quad 10] = 3 + 1$$

A: From here

Q: How did I get here?

Edit Distance

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

Q: How did I get here?

Edit Distance

	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

A: From here

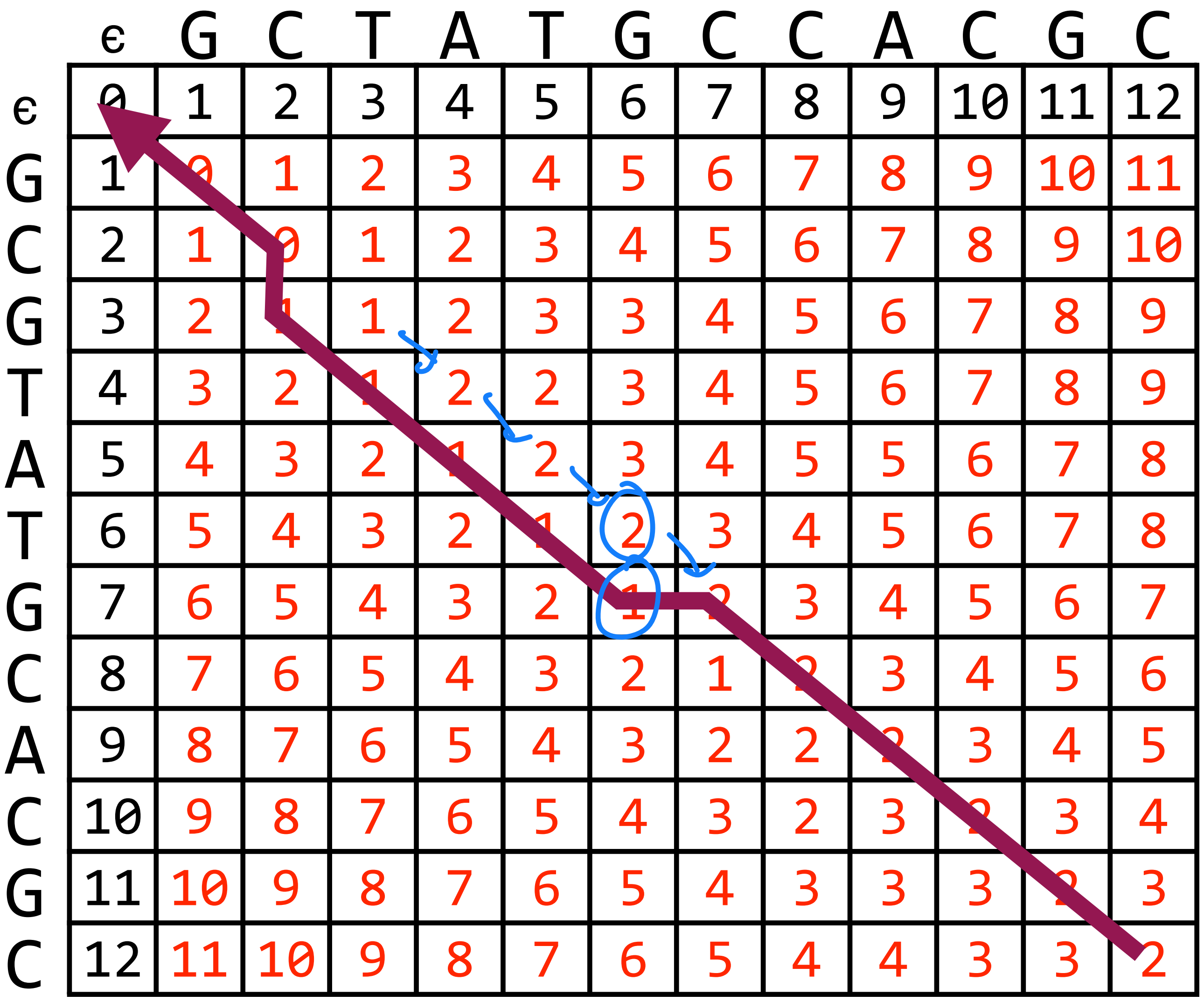
Q: How did I get here?

$$D[9,\ \ 10] = 3 + 1$$

$$D[9,\ \ 9] = 2 + 0$$

$$D[10,\ \ 9] = 3 + 1$$

Edit Distance



	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

Alignment:

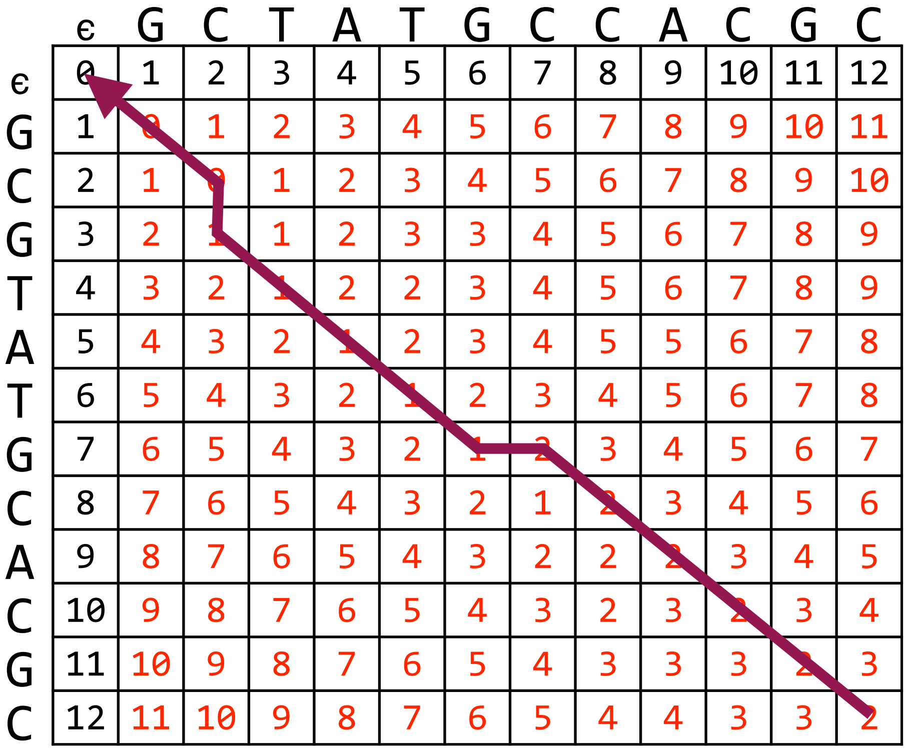
```
GCGTATG-CACGC
|| ||||| |||||
GC-TATGCCACGC
```

MMDMMMMIMMMMM

Edit Distance

Dynamic Programming fills our table with optimal distances

Traceback identifies the optimal *edit string* to convert X to Y



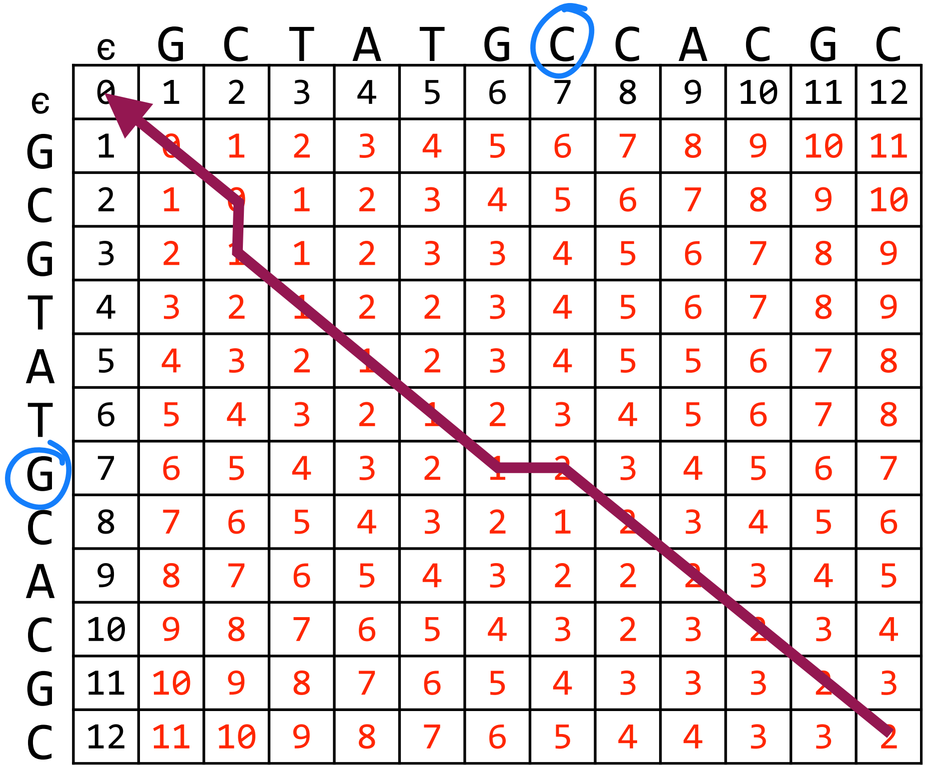
	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

What is our efficiency? ($|X| = m$, $|Y| = n$)

Edit Distance

Dynamic Programming fills our table with optimal distances

Traceback identifies the optimal *edit string* to convert *X* to *Y*



	ϵ	G	C	T	A	T	G	C	C	A	C	G	C
ϵ	0	1	2	3	4	5	6	7	8	9	10	11	12
G	1	0	1	2	3	4	5	6	7	8	9	10	11
C	2	1	0	1	2	3	4	5	6	7	8	9	10
G	3	2	1	1	2	3	3	4	5	6	7	8	9
T	4	3	2	1	2	2	3	4	5	6	7	8	9
A	5	4	3	2	1	2	3	4	5	5	6	7	8
T	6	5	4	3	2	1	2	3	4	5	6	7	8
G	7	6	5	4	3	2	1	2	3	4	5	6	7
C	8	7	6	5	4	3	2	1	2	3	4	5	6
A	9	8	7	6	5	4	3	2	2	2	3	4	5
C	10	9	8	7	6	5	4	3	2	3	2	3	4
G	11	10	9	8	7	6	5	4	3	3	3	2	3
C	12	11	10	9	8	7	6	5	4	4	3	3	2

What is our efficiency? (|*X*| = *m*, |*Y*| = *n*)

Table filling: Filling (m + 1) (n + 1) cells, each requiring constant work, so *O*(*mn*)

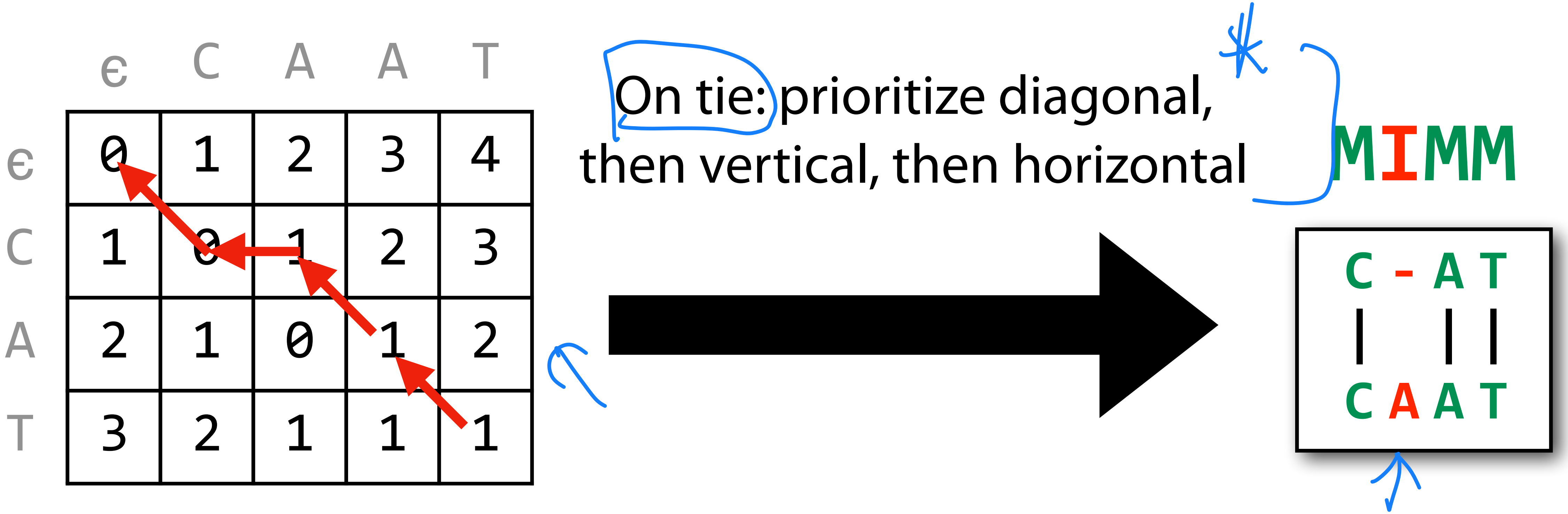
Traceback: Each step goes ⇖, ⇐ or ⇑. Worst case: traceback never moves diagonally, requiring *m* ⇑'s and *n* ⇐'s, so *O*(*m* + *n*)

`string buildEditString(X, Y)`

Input: **`string X:`** Input string X (edits with respect to X)

`string Y:` Input string Y (edits turn X into Y)

Output: **`string`**: An optimal edit string produced by the matrix



	ϵ	C	A	A	T
ϵ	0	1	2	3	4
C	1	0	1	2	3
A	2	1	0	1	2
T	3	2	1	1	1

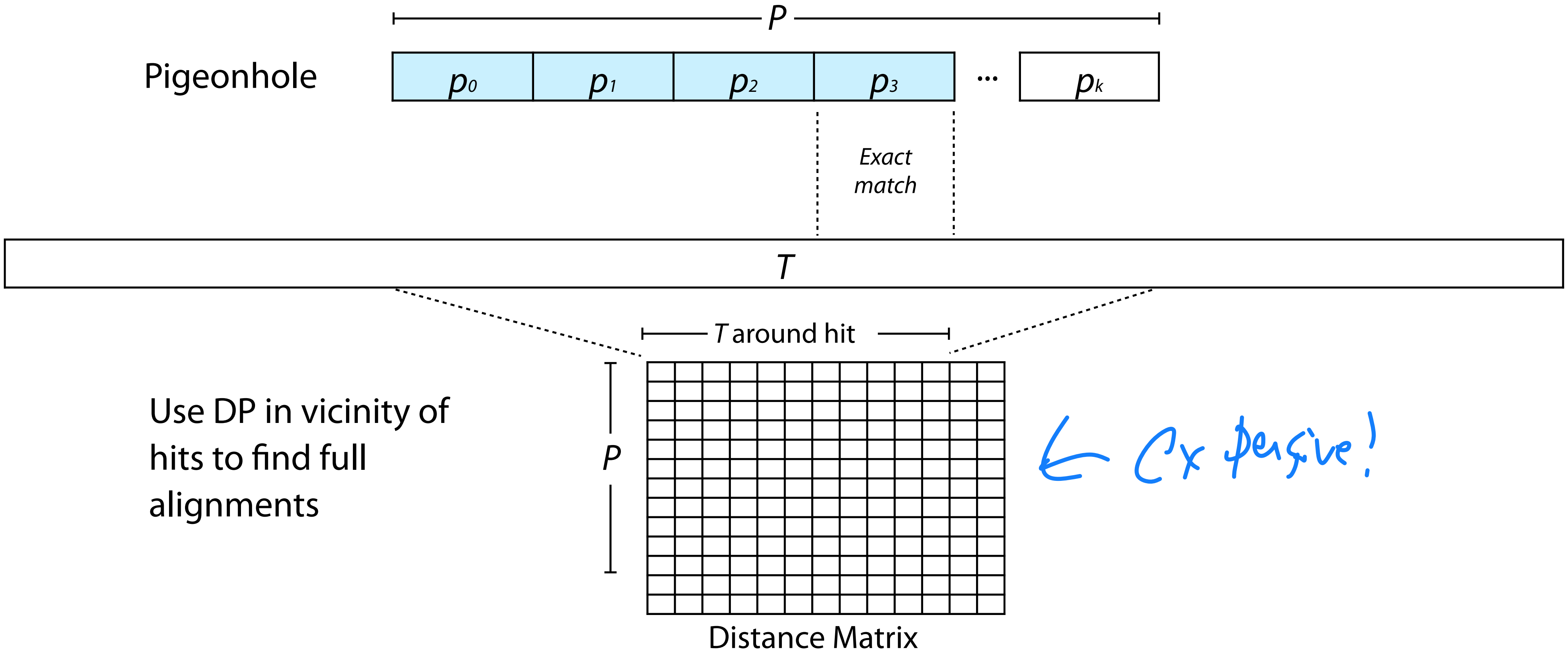
On tie: prioritize diagonal, then vertical, then horizontal

MIMM

```
C - A T
|   | |
C A A T
```

Approximate Pattern Matching

"Seed and extend" works for edit distance too!



Use DP in vicinity of hits to find full alignments

← expensive!