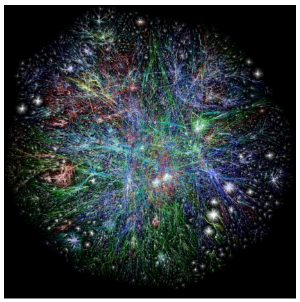
CS 225

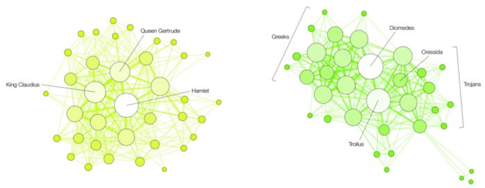
Data Structures

March 29 – Minimum Spanning Tree (Prim)

G Carl Evans

Graphs





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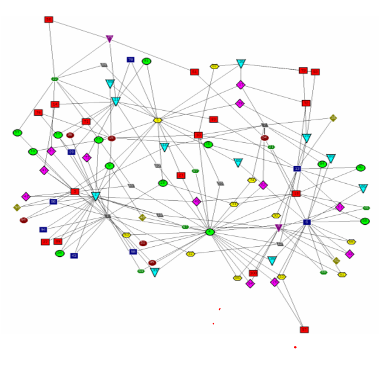
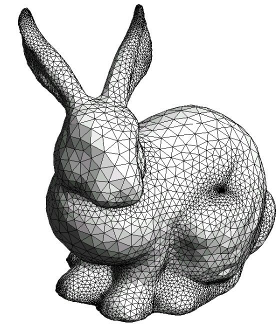
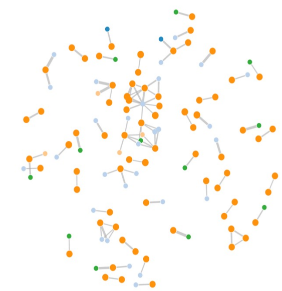
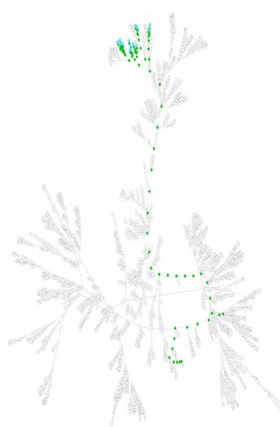


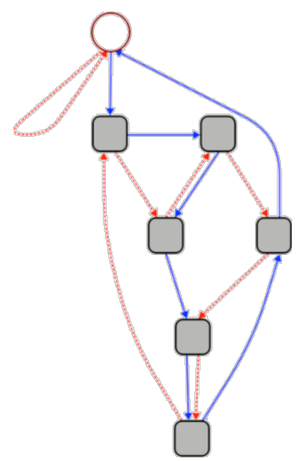
To study all of these structures:

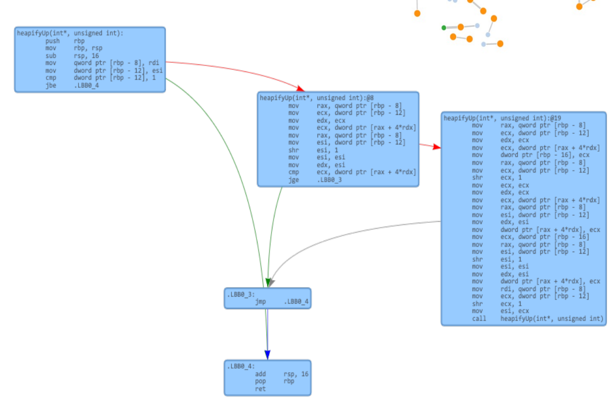
1. A common vocabulary
2. Graph implementations
3. Graph traversals
4. Graph algorithms









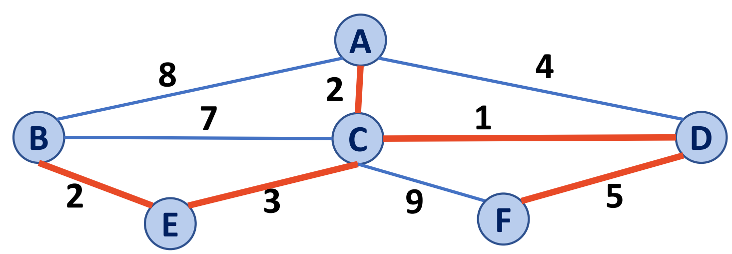


Minimum Spanning Tree Algorithms

Input: Connected, undirected graph **G** with edge weights (unconstrained, but must be additive)

Output: A graph G’ with the following properties:

- G’ is a spanning graph of G
- G’ is a tree (connected, acyclic)
- G’ has a minimal total weight among all spanning trees

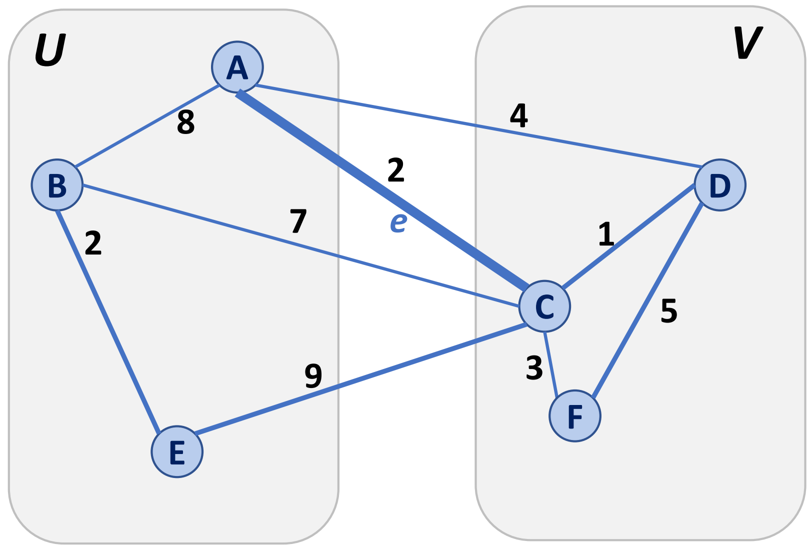


Partition Property

Consider an arbitrary partition of the vertices on **G** into two subsets **U** and **V**.

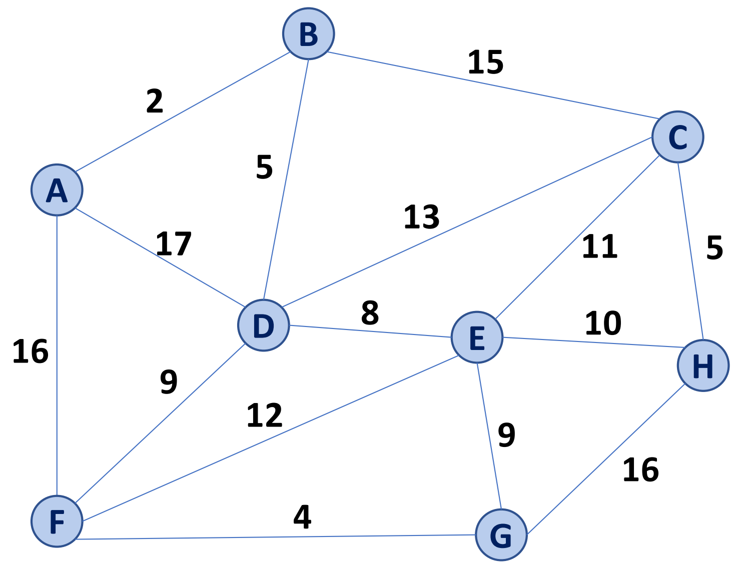
Let **e** be an edge of minimum weight across the partition.

Then **e** is part of some minimum spanning tree.

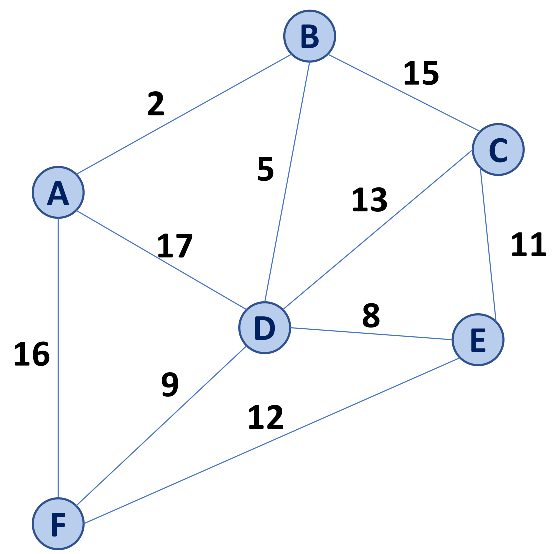


Partition Property

The partition property suggests an algorithm:



Prim’s Algorithm




```
PrimMST(G, s):
  Input: G, Graph;
         s, vertex in G, starting vertex
  Output: T, a minimum spanning tree (MST) of G

  foreach (Vertex v : G):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q   // min distance, defined by d[v]
  Q.buildHeap(G.vertices())
  Graph T           // "labeled set"

  repeat n times:
    Vertex m = Q.removeMin()
    T.add(m)
    foreach (Vertex v : neighbors of m not in T):
      if cost(v, m) < d[v]:
        d[v] = cost(v, m)
        p[v] = m

  return T
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	Adj. Matrix	Adj. List
Heap	over estimate $n + n\log(n) + n^2\log(n)$	$n + n\log(n) + m\log(n)$
Unsorted Array	$n + n^2 + n^2$	$n + n^2 + m$

Prim's Algorithm

Sparse Graph:

Dense Graph:

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```

	Adj. Matrix	Adj. List
Heap	$O(n \lg(n) + n^2 \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array	$O(n^2)$	$O(n^2)$

MST Algorithm Runtime:

We know that MSTs are always run on a minimally connected graph:

$$\mathbf{n-1 \leq m \leq n(n-1) / 2}$$

$$\mathbf{O(n) \leq O(m) \leq O(n^2)}$$

MST Algorithm Runtime:

- Kruskal's Algorithm:

  O(n + m lg(n))

Sparse Graph: $m \sim n$

$O(n + n \log(n))$

Dense Graph: $m \sim n^2$

$O(n + n^2 \log(n))$

- Prim's Algorithm:

  O(n lg(n) + m lg(n))

Sparse Graph:

$O(n \log(n) + n \log(n))$

$\equiv O(n \log(n))$

Dense Graph:

$O(n \log(n) + n^2 \log(n))$

$\equiv O(n^2 \log(n))$

so in both edges same

Suppose I have a new heap:

	Binary Heap	Fibonacci Heap
Remove Min	O(lg(n))	O(lg(n))
Decrease Key	O(lg(n))	O(1)*

What's the updated running time?

$O(n \log n + m)$ with Fibonacci Heap

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```

$O(1)$ not $O(\log(n))$

MST Algorithm Runtimes:

- Kruskal's Algorithm:
  O(m lg(n))
- Prim's Algorithm:
  O(n lg(n) + m lg(n))

Shortest Path